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THE CASE FOR GENERAL MATHEMATICS[1]

By WILLIAM DAVID REEVE
University of Minnesota

I shall not attempt, in this paper, to discredit our traditional methods of teaching algebra in the first year of the high school, followed by plane geometry in the second year, intermediate algebra in the third year, and so on. I say this in spite of the fact that much of our traditional practice and the accompanying results might justify one in so doing. In short, I am not interested in a destructive type of criticism of past methods with a view to setting up new bits of content (or at least reorganized content) and technique of procedure. Certainly, I should not favor a method which would seem to be attempting to force any set program upon the teaching body. The best progress is not made in that way. With many teachers of mathematics, the traditional order of treatment, if not the traditional methods, will prevail. Moreover, this will be true even after much experience and available scientific data may make a trial of some form of reorganized content and methods seem wise and feasible.

There are teachers in this country, however, who not only do not want to be limited to certain set programs, but who covet the opportunity to learn better methods of doing things by actually trying out materials and then observing and measuring results. It is because of such an attitude that the content and methods of teaching of first year algebra have been improved in recent years. A corresponding improvement has been made in the content and methods of teaching plane geometry, and I, for one, am grateful for whatever progress we have made. Teachers pretty generally, have agreed with the improvements referred to above, but they have been rather slow in changing their

[1] Read at the meeting of the National Council of Teachers of Mathematics at Chicago, March 1, 1920.

courses to agree with them. We shall continue to make progress along this line, but we must not expect a large part of our teaching body to assimilate new material very fast. We must be patient and lift our standards and reorganize our courses in such a way as not to be misunderstood. Even such a body as the College Entrance Board may not be expected to be as ready to make as radical changes as certain individual teachers may see fit to suggest.

We have come to believe in scientific methods in education. Those of us who happen to be connected with experimental schools of one kind or another have come to feel more deeply than ever before the responsibility that rests upon our shoulders in making some worth while contribution to our fields. We are not bound to proceed by set aims and objectives alone. We start a method of teaching, measure results, and thus arrive at certain conclusions that finally seem valid and important. After all, may not one seriously ask "What do we mean by set objectives?" Or perhaps, "Do we always secure the best results by emphasizing set objectives?" We are continually reminded by one educator or another that we are not securing these objectives. They even accuse us of not having any objectives at all and in many cases this is no doubt true. For example, the other day a friend of mine described to me an experiment that a research student was carrying on in a certain college of agriculture. He was experimenting in the feeding of hogs. No final objectives had been set up, but by careful observations and measurements he finally obtained some very valuable results to be worked for in the feeding of hogs.

I understand that the Latin teachers, in their national survey, are more concerned with doing things and the measuring results than they are in setting up and defending any set objectives in the teaching of Latin. This, it seems to me, is wise, not because we cannot set up certain objectives, but because the methods of measuring results is more likely to keep one open minded. The only difficulty here lies in not having time and money enough to continue the experiment long enough.

The proof of the pudding is in the eating. The case for general mathematics, so far as I am concerned, rests largely on the fact that it gives results that are highly gratifying. This is

especially true if the work is continued from two to four years in the high school. The movement for a general course in mathematics is certainly not a new idea. John Perry, of England, saw the importance of recognizing the content and method of teaching mathematics many years ago. He put the case as follows: "Great fields of thought are now open which were unknown to the Alexandrian philosophers. If we begin our study as the Alexandrian philosophers did, with their simplest ideas in arithmetic and geometry, we shall get stale before we know much more than they did. If we begin assuming more complex things to be true (although I do not like to assume that in truth any idea is more complex than another) as we have done in arithmetic, as we ought to do in other parts of mathematics without becoming stale we may know of all the modren discoveries. We shall thus get the same intellectual training with more knowledge." He says further, "In these days all men ought to study natural science. Such a study is practicably impossible without a knowledge of higher mathematical methods than that of the mere housekeeper. It must be more than what is called 'knowledge,' it must be mental dexterity, and it must be kept in constant practice if it is not to become rusty, and if men are to remain unafraid of mathematics. As examples of the methods necessary even in the most elementary study of nature I may mention: the use of logarithms in computation; knowledge of and power to manipulate algebraic formulae; the use of squared paper; the methods of the calculus. Dexterity in all of these is easily learned by all young boys. In such practice their brain power develops quite rapidly and they learn with pleasure. I feel sure that such dexterity cannot hinder, and can only further the mathematical study of the exceptionally clever sudent."

"For an advanced study of natural phenomena we need the results of the best study of the greatest mathematicians. To me mathematics is a powerful weapon with which to unlock the mysteries of nature. If a man knows how to use the weapon, that is enough. Let him leave to others, the men who delight in that, the forging of a weapon, the complete study of it. If I can use the weapon, let my study be of another kind—I think of a higher kind—to study the secrets which even an unskilled use of the weapon will reveal to me."

"I have the belief that the study of physical science, and therefore the study of mathematics, by everybody, however poor or however rich, is of the utmost importance to our country, not merely for the knowledge it gives, but for producing the scientific habit of thought, giving to every unit of the population a power to think for itself, and so producing the greatest happiness and giving the greatest strength of all kinds to the nation."

Classroom experience shows that when certain things in mathematics are necessary to the child's progress they should be given him, if need be, without proof. He should be taught how to use them until they become a part of his machinery and later, when he is more mature, he can study further and develop the nature of the machinery he has been using. This is what we all do every day. We simply take on faith a number of things we cannot fully understand. We can thus lead the child into much higher and more powerful mathematics without any ultimate loss. It is well known in our study of mathematics that we are continually making large assumptions where failure to do so would be not only unwise but ridiculous.

Many of us have been spending a whole semester on material that ought to be covered in considerably less time. In this connection I want to suggest that great economy in time could be achieved if only we would determine, scientifically, how long it takes to teach a certain topic satisfactorily. And by satisfactorily we should mean "to teach the topic to the degree of perfection indicated by some standard agreed upon." For example, how long does it take to teach, say a normal group, or even a certain kind of selected group, to learn how to factor the difference of two squares so that nobody in the group would make more than one error out of fifteen possible cases?

We cannot expect complete learning of a topic the first time over, as some writers would have us believe. Such a belief is contrary to the facts gained by class room experience. We should adopt some good spiral plan that will give good results.

Then too, what harm can come from omitting a great deal of the material that we have been teaching in algebra and geometry and taking for granted a great deal more that we have been forced to try to prove with little or no success? There is not a person today who cannot point out some phase of his math-

ematical training that has been of little real service compared to what might have been expected considering the time spent in acquiring that particular experience. The old fashioned geometrical method of proving the Pythagorean Theorem can be replaced by a simpler and shorter algebraic method known to every competent teacher. In fact, a great deal of Euclidean geometry can be proved by simple algebra if we permit unification of the two subjects.

The crowded condition of the freshman classes in some of our colleges and universities suggest further the advisability of furnishing more opportunities for a further pursuance of mathematics in the high school than is often the case. This may or may not lead eventually to a more serious consideration of the Junior College in many places. In any case, the plan, if followed, will enable the colleges to secure a better trained group of entering students and to devote more of their time to some of the advanced aspects of the subject. The trouble has been that we have spent so much time on the elementary phases of the subject trying to make sure that the child learns everything before he goes on, that we are not able to give him much in advance of what his ancestors had. As a result, a number of our colleges and universities are teaching little in the first two years beyond high school work. A great deal of the material we have to learn has never been of value to us either in mathematics itself or in the allied fields. Let us teach not so much mathematics, but more about mathematics.

My proposal is that wherever the conditions permit we arrange our mathematics so that we shall not have courses in algebra, in geometry, or in trigonometry as such, but a definitely arranged and psychologically ordered course in mathematics. The plan which we are now formulating will give the student who desires it a four year course in mathematics in the high school plus one or one and one half years of college work. It is also hoped that the course will be more compact, will involve less waste, develop more power, and produce even better results than we have been getting from traditional methods. The content of this course will be algebra, plane and solid geometry, trigonometry, most of the analytic geometry, and the fundamental elements of the calculus.

There is nothing essentially new in the plan of teaching algebra and geometry together. The best teaching talent in mathematics the world over, long ago recognized the importance of emphasizing the relation between algebra and geometry and the advisability of teaching them together. The movement to unify the two subjects has been vigorously opposed by those who seriously insist that such unification will break up the logically ordered system of Euclidean geometry and give rise to a series of unrelated ideas without unity or natural sequence. But if we are ever to cut the "ancestral process" short enough to enable us to give the boys and girls anything beyond what we ourselves have had, we must omit a great deal that was formerly taught and reorganize what is left in a more psychological way. We can still plan courses in algebra and geometry, as such, for the special needs and the special cases where such procedure seems necessary and wise.

The recent work and reports of the National Committee on Mathematical Requirements have furnished us with much valuable material and their final report will furnish us a basis for much further research work in the content and teaching technique concerning high school mathematics. But we must make sure that the good work of this committee is carried on.

The purpose of a general mathematics course in the high school should be to furnish a basis for a modern scholarly course in elementary mathematics that will give such careful training in power and appreciation as well informed citizens of our democracy ought to possess. And further, to arrange the material so that the boy or girl who is forced to leave the high school at the end of one or two years may nevertheless get a better understanding and appreciation of some of the finer things of life so that he may enjoy himself as he goes along. For the student that remains, it is believed a general course will save at least a year and that his ultimate conception of the entire field will be more thorough and fundamental.

In the traditional high school course, algebra is taught in the first year, geometry in the second, intermediate algebra and solid geometry or trigonometry in the third. It is very unusual that any mathematics is offered in the fourth year. Until recently these subjects were taught in water tight compartments so that

when the student studied algebra he felt that he was through with it, and so with plane geometry. He has not, as a rule, been taught to see how one subject may be made to reinforce and supplement the other. This artificial pigeon-holing of the subject matter has been a practice that good and well trained teachers of mathematics have never observed, but the rank and file have been very much hedged in by the traditional practice of treating the topics separately. Does it not seem rather a matter of tradition only that we have kept the algebra, the geometry, and the trigonometry in strictly parallel lines of treatment? Does this mean that we shall always sacrifice the psychological for what may often be considered the more logical order of treatment? "Teachers in lower grades have never realized that the union of logic and space studies deprived them of one of their most natural subjects of instruction, namely, form study. The logical statement of the principles of geometry has blinded modern as well as medieval teachers to the true worth of this subject for younger pupils." There is no doubt that the traditional tandem treatment of algebra first, geometry second, and so on, gives rise to a great deal more waste than many teachers realize, and it does not show the intermingling of the subjects as it should. A general mathematics course in the high school will show us more clearly how each subject is reinforced and made clearer and more helpful by the other.

The organizing and unifying principle of the general mathematics course in my own school is the idea of the functional relation—the dependence of one quantity upon another. In the first year we make the function the background of the course and we make the simpler truths and constructions of geometry help to rationalize some of the more formal aspects of the algebra and later to furnish exercises for algebraic applications. This is done in various ways, but the function concept, either implicitly or explicitly, dominant throughout, helps to lend concreteness and coherence to the subject. Excessive formalism is greatly reduced and the emphasis is placed upon the function, the equation, the formula, and the graph. Enough time is saved to permit us to furnish more illustrations and applications of principles and to introduce new and more important material.

Instead of waiting until the second year and trying to crowd all of the difficulties of demonstrative geometry into that year, we introduce much intuitional geometry in the first year, followed in some cases, by simple demonstrative geometry only where it comes naturally and easily. In this way many of the relations are taught inductively by experiment and by measurement, to be followed later by more formal proofs. The children who have geometry in this way do not start the geometry work in the second year as if it were an entirely new subject, as has often been the case.

In the next place, our traditional methods have delayed the teaching of much that is interesting and valuable in the secondary field. In this respect many of the English and continental schools of Europe are far in advance of us. The elementary ideas of numerical trigonometry, in many respects very simple, have been omitted altogether from most secondary work. They are as easy for a freshman in high school as anything else, once he understands similar triangles.

In the second year we take up the more logical and rigorous methods of demonstrative geometry as the central theme, but we keep the algebra before the student constantly by means of algebraic applications. In addition, we take up many of the facts of solid geometry at the points where their analogy is most natural and easy.

The work in trigonometry is carried on by giving more advanced problems in the solution of right triangles by logarithms, some work in proving simple identities, and some introduction into the solution of oblique triangles, where the law of sines applies. A great deal of emphasis in the second year is given to the different methods of attack and habits of studying. The student is made familiar with inductive, deductive, analytic, synthetic, and indirect methods of proof. By the end of the year we expect our capable pupils to do a high type of work.

In the third year the geometry is not so prominent and comes in again in the role of a helper to the algebra and trigonometry, although solid geometry is finished in this year's work. The algebra and trigonometry are related within topics as near as possible; e. g., when we are treating algebraic equations in one

unknown we treat trigonometric equations in one unknown and try to make clear the likenesses and differences in the nature of the solutions. We have ample opportunity to do a great deal with the simpler elements of analytic geometry without any loss of time so far as we can see.

In the fourth year we hope to unify the college algebra and the analytic geometry with some of the elements of the calculus in a better way than we have been able to do it so far by giving the earlier courses more compactness and better treatment. It goes without saying that not every student would be expected to study mathematics through four or even through three years, but it is interesting to note that where such opportunity is offered there are always students eager to register for the various courses though they are not required. At present we have fifteen seniors out of a class of fifty taking their fourth year of high school mathematics.

Our methods of teaching mathematics also need to be improved. The knowledge we now possess of individual differences in ability should make the study of mathematics a kind of laboratory course, in which more effective work can be done because the material can be better fitted to the individual pupil's needs. Such an arrangement of material decreases the need for so many reviews because each subject is kept in more or less constant use. As a result there is a gain in mathematical power and less need of home study. In the first year it is even possible to get along with little or no homework.

In order to give an idea of what some of the pupils taught in general mathematics course think of the method, the writer is quoting below a theme written by a freshman in one of his classes. The theme was handed in to an English teacher as a regular daily exercise. The title was "Mathematics, My Favorite Study." It ran as follows: "I think the first year of math is one of the best studies I have ever had for the following reasons: first, it is interesting; second, it is all, so far, based on one general idea, the equation; third, if absent, the work is easily made up. I can work for hours and hours at a time doing my "math" homework, because I like the study and it comes easy for me. I never count mathematics as a study; I think more of it as a sport,

like gym, not because I "rough house", but the time passes so quickly and I learn so many new things. In this way it is interesting.

"At the first of the year we thought of the equation as an expression of balance, as scales. Then we used the equation for angles, verbal and motion problems, graphs and finally for parallel lines. This brings up my interest, seeing all the different ways of using the same thing, and seeing that the equation can be used for almost all problems.

"As to being absent, that is easy. You have your book and you can go right ahead and do the problems as they are fully explained. If you can't get it this way, go to the study class when you come back to school, where you can get all the help you want, direct from the teachers. Can't you see that mathematics is a wonderful study, and why I like the first year of it so well?"

It is for the class of students who love mathematics or show unusual ability in it that I should want especially, to see the opportunity given to go on with their work in such a general course. They are the ones who will be the future teachers or research students in the subject.

At present I am not prepared to say what the colleges might do to adapt their courses to the students who finish the kind of course outlined above. That is largely a problem for the colleges to solve; but there are some things that will add greatly to the chance of an early agreement on a course. In the first place, the high school teachers should get together and formulate more definitely than heretofore minimum courses for each year of the high school, setting up certain standards of attainment. Secondly, the college teachers should familiarize themselves with these courses and, if they are satisfactory, they should build their college courses upon the high school courses as far as possible. In this connection, I have never understood why the general mathematics courses in the colleges have not been more successful. Thirdly, we ought to have frequent visitations back and forth wherever possible to enable those on both sides to keep in mind just what is to be expected as a final outcome and how much is being done on each side toward making a proper contribution. Then, if we study our habits, and improve our technique, we shall get results that are worth while and mathematics

will maintain the dignity which it has so long held in the curriculum. There has been an enormous lot of time wasted by repetition and so-called reviews that do not deserve the name. And the fact that the college courses do not fit on properly to the high school courses has led to a still further loss of many a student's time. Then too, a great many colleges and universities fail to provide courses at the proper time for students who are ready and eager to go on and often loss of time and gaps in instruction appear. It will be argued, of course, that proper adjustments are impossible because of administrative difficulties, but it is not hard to conceive of a little better situation than we often find.

It may be of interest here to say that of all our University high school students who have had at least two years of work in general mathematics, and who have subsequently taken further courses in the University, 13.2% have received marks of A; 31.6% marks of B; 34.2% marks of C; 13.2 marks of D; and 7.8% marks of F. Only two students have failed and one of these was one of our honor students at graduation and the other stood at the bottom of the class.

Moreover, of these who did not have at least two years of work in general mathematics, i. e. those who had their mathematics before they came to us, and who have taken further mathematics in the University, no one received a mark of A; 11.7% received B; 25.6% received C; 39.5% received D; and 23.2% were given a mark of failure. Of course I should not claim that these results establish the claims for a general mathematics course in the high school, but they are interesting.

ERRORS IN COMPUTATIONS AND THE ROUNDED NUMBER[1]

By PROFESSOR HARRIS RICE,
Worcester Polytechnic Institute

I believe it is now quite generally felt among teachers of mathematics that it is their duty to give the boy and girl such mathematical instruction as may be of use to them in their daily life after leaving the high school. This feeling has given rise to some of the modern ideas as to what should be taught the boy and girl who does not go beyond the high school. Therefore they say properly that such students should be given a little trigonometry, a little differential calculus, a little integral calculus, not very much of these topics, but just enough to give the student an idea of what these subjects are and for what they are used. Such modern ideas seem to the writer very desirable. Mathematics has too long been held up as a very mysterious subject, something which the boys and girls of the high school have been very apt to dread—and if mention should be made of such a terrible subject as calculus, they would give up in despair.

The object of the present paper is not to cry down the introduction of such material into the high school curriculum but rather to urge that more emphasis be put upon some subject matter already there. I say more emphasis, but I am almost tempted to say "some mention should be made of it," for I find that in many secondary schools this topic which I have in mind is not touched upon at all. This condition exists not because the subject matter is difficult to grasp but because the rank and file of teachers are not familiar with it. Text book writers have made this subject so difficult that a great many teachers think it is way beyond them. It is my intention in the present paper to lead mathematics teachers to feel differently towards this very important part of mathematical instruction.

I have chosen for the title of my paper, "Errors in Computations and the Rounded Number." This does not mean that I shall give a discussion of the theory of errors nor that I shall discuss the question of accuracy from the differential point of

[1] A paper read before The Association of Teachers of Mathematics in New England, Boston, Mass., May 6, 1922.

view as is done in most text books. I have avoided such a discussion and also the differential notation on the grounds that such a method of attack would be of no use to the secondary school pupil. I do this not because I doubt for a moment the ability of my readers to follow such a discussion but because I have a certain objective in mind; to make this subject seem not only desirable for secondary school work but a necessary part of any secondary school course in mathematics.

Among all the various classes into which numbers have been divided since the days of the early mathematicians, let us consider for a moment the two classes known as counting and measurement numbers. A counting number is the number given in answer to the question, "How many?" In this class of numbers belongs zero and positive integers. Negative numbers and fractions have no place in this system. For instance, in answer to the question: How many were present at the meeting? we would never give such answers as—9 or $5\frac{1}{2}$. Our answer would be a very definite positive number as 15, 25, 0, etc.

Perhaps you are now ready to say that a measurement number answers like questions. It counts up the number of inches or feet or whatever the unit of measurement is, you may say, and therefore I have no right to put measurement numbers into a different class from counting numbers. I admit that they do count up the number of units of measurement, but let us see if there really is not a difference after all between what I have called a counting number and what you have called a counting number.

Suppose I ask you to measure the length of a certain table, and you say after performing the task that the length is 3 ft. $7\frac{3}{8}$ in. It appears that you have scored against me for you have given me in your answer a fraction, something which I said could not exist in my system of counting numbers. "Very well," you may say, "I inadvertently gave you my answer in that form. I might just as well have said that the unit of measure is one eighth and I find that there are 347 units in that length." This answer apparently satisfies the definition I have given for a counting number, but I still insist that there are reasons why it cannot be included in the class of counting numbers.

I can now point out the fundamental difference between a counting number and a measurement number. When you tell me that there are 15 people at the meeting, you mean that there are exactly 15, no more or no less. But when you say that the length of the table is 3 ft. 7⅜ in., can you hold up your right hand and swear that your measurement is so exact that the true length is not .001 in. more or less than your 3 ft. 7⅜ in.? Of course you cannot do so. What you have given represents the length of the table to the nearest eighth of an inch and that is all. You know that it is nearer 3 ft. 7⅜ in. than it is 3 ft. 7²⁄₈ in. or 3 ft. 7⁴⁄₈ in., but beyond that you cannot go. You may select measuring sticks with finer degrees of division and measure your table to the nearest 16th, 32nd, 64th of an inch, but no matter how minute the division you can never say with absolute certainty that you have recorded the exact length of the table. Such a number is what I call a true measurement number. While it does count up, it counts up in an entirely different way than a counting number does.

All measurement numbers are thus seen to be approximate. They represent a length to the nearest fraction of a unit, the degree of approximation depending upon the care we take in measuring the length, the accuracy of our measuring instrument, and the size of the smallest unit on the instrument employed. In any case they are approximate or, as we say, represent the length in round numbers. Such numbers are therefore called rounded numbers. Any number which is used as an approximation is called a rounded number. The radius of the earth in round numbers is 4000 miles.

Since a rounded number is any number which is used as an approximation in place of the true value, the class of rounded numbers is not restricted to measurement numbers alone. While all measurement numbers are rounded numbers, all rounded numbers are not measurement numbers. Nearly all decimal fractions are rounded numbers. When .33 is used for ⅓ we approximate the value of ⅓. We say that ⅓ in round numbers is .33. Trigonometric functions, logarithms, irrational numbers are rounded numbers. The value of π in round numbers is 22/7.

We say that the value in round numbers of ⅓ to two decimal places is .33; the sine of 45 degrees to four decimal places is

.7071; the logarithm of 3 to seven decimal places is .4771213; the value of π to fifteen decimal places is 3.141592653589793.

Example 1. *Rounded Numbers.*

$$
\begin{aligned}
1/3 &= .33 \\
\sin 45^\circ &= .7071 \\
\log 3 &= .4771213 \\
\pi &= 3.14159\ 26535\ 87993
\end{aligned}
$$

These are all rounded numbers given to so many decimal places. Such numbers give rise to our familiar 4, 5, 7, 10 place tables.

In each of the above cases I have given the number rounded to so many decimal places. It is much more convenient in general to speak of a number as being rounded to so many significant figures. We would then say that we have given 1/3 to two significant figures; the sine of 45 degrees to four significant figures; the logarithm of 3 to seven significant figures; the value of π to sixteen, not fifteen significant figures.

What are significant figures? The digits 1 to 9 inclusive are always significant. A zero may or may not be significant, depending upon its position in the number. If a number is composed of the digits 1 to 9 inclusive only, then the number of significant figures is determined by counting the number of digits present. If a zero occurs in the number, it is necessary first to determine whether or not it is significant. The significant digit lying on the extreme left is called the head digit as it is the most important digit in the number.

If the zero lies between two digits known to be significant, then it is significant also. For instance, (see example 2), 72.03 is a number of four significant figures; 200.24 is one of five.

Example 2. *Significant Zeroes.*

72.03	200.24
00.00241	.00241

$$
\begin{aligned}
1/4 &= .2500 \\
1/3 &\neq .330 \\
&2.340
\end{aligned}
$$

If the zero lies to the left of all digits known to be significant, it is not significant, and should never be used in this way except

between the decimal point and the first significant digit as in the case of decimal fractions. The head digit can never be zero. Thus we should never write 00.00241, but instead .00241. Neither of these two zeroes is significant. They are simply fillers to locate the decimal point in its proper place. The number .00241 is thus a number of three significant figures.

If the zero lies to the right of all digits known to be significant, then it may or it may not be significant. If I say that ¼ to four decimal places is .2500, both zeroes are significant and we have ¼ expressed in round numbers to four significant figures; if I say that ⅓ in round numbers is .330, then the zero employed is not significant, and I have expressed the value of ⅓ to only two significant figures; if I have given the number 2.340, I say that I do not know whether the zero is significant or not, and so I call it a doubtful zero.

By what means am I able to say that a right hand zero is significant, not significant, or doubtful? The answer to this question lies in the definition of a significant figure. You may have noticed that I have not yet defined what I mean by a significant figure, simply contenting myself with telling you what figures were significant in each of several cases. I have done this purposely, and I am now sure that you can define it yourself. It is simply a figure which gives some significance to the number, some added significance as regards the accuracy. You would feel much more informed in regard to the decimal value of ¼ if I should tell you that it was .25 than if I should tell you it was .2. In one case I have given the value of ¼ to two significant figures, in the other to one. In other words, a significant figure adds to the accuracy of the number. And this is the test for a right hand zero: ask yourself this question: If I knew that number more accurately would that zero disappear and some other digit take its place? If your answer is no, the zero is significant; if your answer is yes, the zero is not significant; if you cannot answer the question, the zero is doubtful.

Apply this test to each of the above cases. The value of ¼ is exactly .2500. There is no question about that, so that is why these two zeroes are significant. As a matter of fact, every zero you put on after the 5 would be significant for you could not replace them by more accurate digits. In the case of .330 stand-

ing for $\frac{1}{3}$, we know that the zero is not correct, for $\frac{1}{3}$ to three decimal places is .333. Hence the zero disappears when we know the number more accurately and so is not significant. In the last case, that of the number 2.340, we do not know what the number represents and so cannot say whether we should replace it by some other digit or not. Hence it is doubtful. We should never allow a zero to appear at the right of the digits 1 to 9 inclusive unless it belongs there by absolute right or unless it is put there to fix the decimal point. When we say that the radius of the earth is 4,000 miles, all three zeroes are not significant, for the radius is known to be 3,959 miles, but in giving the radius of the earth in round numbers to one significant figure these zeroes have to be put there to fix the decimal point. They determine the size of the number but not the accuracy. If on the other hand you tell me that a certain measurement yields for a result 1.500 cms., I assume that you have measured so carefully that the number of hundredths and thousandths has been determined to be zero. If not, then these zeroes should not be given, for they give a false accuracy to your result.

A very simple way to determine whether or not a zero is significant is to change the number to standard form. To change a number to standard form, place the decimal point after the head digit and offset by multiplying by the proper power of ten. (See example 3). If the zero in the original number still remains in the number when expressed in standard form, the zero is significant; if it does not so appear, it is not significant.

Example 3. *Standard Form.*

$$324.2 = 3.242 \times 10^2$$
$$.000241 = 2.41 \times 10^{-4}$$

So much by way of introduction. I have taken the time I have for this preliminary discussion for, in order that you may follow the methods of computation I am going to describe, it is necessary that you should thoroly understand what is meant by the terms ***rounded numbers, significant figures,*** and ***standard form.***

We have seen that a rounded number is an approximation. Hence it does not represent the true value of the quantity in-

volved. The difference between the exact value of the quantity and the value represented by the rounded number is called the absolute error. The relative error is defined to be the ratio of the absolute error to the exact value. Relative error being a ratio, is an abstract number and is often expressed in per cent. If the diagonal of a square 10 in. on a side be measured and found to be 14.1 in., the absolute error is less than .1 of an inch. The relative error is less than .71%. The error expressed as a percentage certainly means a great deal more to us.

I will now take up the fundamental processes of computation with rounded numbers. No general rules can be given save the one included in the general statement: The result of a rounded computation cannot be more accurate than the data. This means that all results should be rounded off to the same number of significant figures as is contained in the data. I shall show that this rule cannot be followed blindly, as there are exceptions. After all is said and done we employ nothing but common sense, and it is by the application of common sense alone to all of our problems that we may hope to solve our difficulties rather than by the application of any formula.

To round off a number we simply drop the superfluous figures, increasing the last digit by 1 if a 6, 7, 8, 9 is dropped; leaving it the same if a 1, 2, 3, 4 is dropped. If a 5 is dropped we want to know whether it is a 5 plus, a 5 minus, or just a 5. Suppose I have the number 23.454. See example 4. This number expressed to four significant figures would be 23.45, and I

Example 4. *Rounding Off.*

23.454	28.248
23.45	28.25
23.5	28.2

would call the 5 a 5 plus. Expressed to three significant figures the number would be 23.5. If on the other hand I have the number 28.248 and round it off to 4 significant figures, I get 28.25. This 5 I call a 5 minus, and would call this number to three significant figures 28.2. When a number ends with the digit 5 and there is no way to determine whether it is a 5 plus or a 5 minus as we did above, this rule is followed: Increase the preceding digit if it is odd; leave it unchanged if it is even.

The reason for this rule is that it has been found that in the course of a large number of computations in a single problem, the number of times you increase the last digit by this rule is just about offset by the number of times you leave it unchanged.

Addition. When two or more rounded numbers are added together, it is clear that the errors will add together also. That is, the error of the sum will be the sum of the errors. Let us add the numbers 26.21, 3.26, .28, and 3.81. See example 5, (*a*).

Example 5. *Rounded Addition.*

(*a*)	26.21	.005	26.	.5	(*b*)	32.621	32.621
	3.26	.005	3.3	.05		.02841	.0284
	.28	.005	.28	.005		5.287	5.287
	3.81	.005	3.8	.05		.0002814	.0003
	33.56	.02	33.38	.6			37.9367
							37.937

Since the error in each number may be as large as .005, the error in the sum may be as large as .02. In other words, the true sum in (*a*) will lie between 33.54 and 33.58.

You have already doubtless observed that I have in my very first example disobeyed the general rule in regard to rounding off implied earlier in the paper, viz. to round off all numbers and the result to the same number of significant figures. Suppose we did follow this rule in this example. We would then have a total error of .6 possible.

If you look carefully at the columns of figures, I think no explanation is needed why the process of addition furnishes an exception to the general rule. I think you will all agree that such a rounding off process would be absurd, for while the above sum is in error at most by an amount equal to .02, the error in the sum, if the other method is used, may be as large as .6. Hence the sum would lie between 32 and 34. I say that such an error would be absurd for we know that the true sum to 3 significant figures is 33.5.

The only general rule I can give applicable to the addition of rounded numbers is the following: Round off all given numbers to be added so that there will not be more than one broken column at the right and then round off the sum so that the last

figure in the sum comes in the last unbroken column. See (b) in example 5.

Subtraction. I shall pass over the subject of subtraction of rounded numbers with the single statement that all that was said in regard to the addition of rounded numbers applies to the subtraction of rounded numbers as well.

Multiplication. Suppose I wish to multiply the two rounded numbers a and b together. Call the error in a x, the error in b y. Then

$$(a + x)\ (b + y) = ab + ay + bx + xy$$

Since ab represents the product of the numbers a and b, the error in the product due to the errors x and y in a and b respectively must be $bx + ay + xy$. Under normal circumstances the errors x and y will be small. Hence the product xy will be very small so that it will be sufficient for our purposes to say that the error in the product is $bx + ay$.

Now suppose $a = 22.5$ and $b = 34.26$. Then x may be as large as .05 and y may be as large as .005, and

$$ay + bx = .1125 + 1.7130 = 1.8380 = 2.$$

In example 6 we have at the left the work of forming the product of these two numbers in the old-fashioned way. The third figure of our result is doubtful since the error may be as large as 2. Hence if we round off the result to three significant figures, 771, we have the product to as many significant figures as we have any right to keep. Note that this number of significant figures agrees with the smallest number of significant figures found in the original data. When two numbers are multiplied together, the result can never be accurate to a greater number of significant figures than is determined by the smallest number in the data. Hence we may save ourselves labor by rounding off both numbers to the same number of significant figures before we multiply. See the middle column of example 6. The result now is 772; to be sure, it is not 771, which we obtained before, but we have shown that the product can lie anywhere between 769 and 773.

The labor involved in such a multiplication may be still further reduced by arranging the work as is shown in the right hand column of example 6. This method may be explained thus: Round off both numbers to the same number of signifi-

cant figures and then add a zero to the multiplicand, place one under the other and draw a vertical line at the right of the last figure. Multiply the multiplicand by the head digit of the multiplier, placing the first figure next to the vertical line. Then round off the multiplicand to one less significant figure and multiply the result by the second digit in the multiplier, placing the first digit of the partial product next to the line. Repeat this process until all digits of the multiplier have been used. Then add in the usual manner and round off the result to the proper number of significant figures. No attempt should be made to place the decimal point by the usual methods. Either determine where it comes by inspection or by changing to standard form.

Example 6. Rounded Multiplication.

```
   22.5        22.5       2350 |
  34.26        34.3        343 |
 ------       -----      ----- |
   1350         675       6750 |
   450         900         900 |
  900         675           69 |
 ------       -----      ----- |
 675          77175       7719 |
 ------       772.        772. |
 770850                        |
 771.                          |
```

The reduction of labor is shown to better advantage perhaps in example 7. Here we multiply 24314 by .00028197. The error in the product may be as large as

$$.5 \times .00028197 + .000000005 \times 24314 = .00026$$

Hence the error comes in the fourth decimal place or, as we see from the table, in the fifth significant figure, as it should, to agree with our former statements.

Example 7. Rounded Multiplication.

```
     24314      243140 |
 .00028197       28197 |
 ---------      ------ |
    170198      486280 |
   218826       194512 |
   24314          2431 |
 194512           2187 |
 48628             168 |
 ---------      ------ |
 685581858      685578 |
    6.8558      6.8558 |
```

Division. Let a and b be two rounded numbers. The question we propose to answer is: What is the error in the quotient obtained when we divide a by b? As before, let x be the greatest error in a, and y the greatest error in b. Then in place of simply a/b what we really have is $(a + x)/(b + y)$, and if we denote the error in the quotient by z we have

$$(a + x)/(b + y) = a/b + z$$

Solving for z we have

$$z = (a + x)/(b + y) - a/b = (bx - ay)/b(b + y)$$

The quantity z is thus seen to be $(bx - ay)/b(b + y)$. Those of you who are familiar with differentials recognize at once the differential form of the expression for z. Practically all text books use the differential as the error. But in my attempt to treat this subject from the point of view of the high school boy or girl I have obtained the result without reference to the differential notation.

In the expression for z, the denominator consists of b^2 and by. Since y is small compared with b, by will be small compared with b^2 and may be neglected entirely. Hence the greatest error in the quotient is approximately $(bx - ay)/b^2$ or rather $(bx + ay)/b^2$ to allow for the error in a being in the other direction.

Let us now use this formula to determine how many significant figures are allowable in the quotient as compared with the number of significant figures in the given numbers.

Let $a = 22.341$ and $b = .367148$

Then $x = .0005$ and $y = .0000005$

and $(bx + ay)/b^2 = .00019474/.13480 = .001$

Hence the error may be large enough to affect the third decimal place of the quotient. Let us divide a by b according to our usual method. See example 8. There is no use in carrying the work any further for we have seen that the third decimal place is in doubt.

What are our conclusions? The quotient cannot be relied upon beyond five significant figures. This number of significant figures agrees with the smallest number of significant figures found in either of the two given numbers. Hence the rule: When two numbers known to be rounded numbers are to be divided, equalize the number of significant figures in each by rounding off the numbers.

The labor of dividing two such numbers will be somewhat shortened if the following method is used, only when this method is used, the precaution must be taken of keeping one extra figure in the dividend when the head digit of the dividend is less than the head digit of the divisor. This must be done in order that the short method will yield a quotient containing the proper number of significant figures. The second column in example 8 illustrates this method. Note the zero added to the dividend for the reason just explained. If you compare the result obtained by this method with the one obtained earlier, you will see that the difference between the two is within the limit of the variation allowed. Hence the result is just as accurate and the labor involved is very much less when this method is used.

The method I think is self explanatory. Instead of adding zeroes to the dividend I have rounded off the divisor one digit at a time after each partial quotient. When zeroes are added to the dividend as in the old-fashioned method, there are assumed a greater number of significant figures in the dividend than are given. There is just as much of an assumption then in the long method as there is in the short method.

Example 8. Rounded Division.

.367148| 22.34100000 |60.848	47
22 02888	36715| 223410 |60849
3121200	220290
2937184	3120
1740160	2936
1468592	184
281568	148
	36

Square. Let us next turn our attention to the error in a square. Again let a be rounded, the greatest error in a being x. Then

$$(a+x)^2 = a^2 + 2ax + x^2$$

Hence the error in the square may be as large as $2ax + x^2$. Since x is small compared with a, x^2 is small compared with $2ax$, and the error in the square is usually taken to be $2ax$. Since this is simply the product of two numbers which we have already

considered in detail. I will not take your time by applying this formula to a numerical case.

Square Root. Using the same notation as before, we may say

$$a + x = \text{(very nearly) } a + x + x^2/4a$$

since the last term is small compared with a. Then

$$\sqrt{a + x} = a + x/2a$$

so that the error in the square root is $x/2a$.

Let $a = 1681$. Then $x = .5$ and

$$x/2a = .5/82 = .005$$

Hence the square root of 1681 could be found to five significant figures.

Let us take another case. Let $a = 11.56$. Then $x = .005$ and

$$x/2a = .005/6.8 = .0005$$

Hence again we would be permitted to obtain a result containing one more significant figure than the original number contains. That the square root should contain the same number of significant figures as the original number is not at all surprising when we remember that the two numbers whose product is a number of four significant figures must contain four significant figures themselves. But for the square root to be allowed one more significant figure then the number contained in the original number I admit makes us pause and think. A moment's thought will show that such a result is reasonable, however. When we multiply we increase the error; when we take a square root we decrease the error.

Let me again call attention to my purpose in this paper, that of presenting the subject in such a way that it may seem not only desirable but also quite necessary to include this subject in the course in secondary mathematics. It is the high school boy and girl who needs these processes. Introduce significant figures and rounded numbers in connection with mensuration problems. Show your students how absurd it is to keep nine or ten figures in a result which is reliable to only a third of that number of figures. Make them see that it is very largely a matter of common sense. As a final word, let me quote the following taken from a book by Professor Ransom: "The rounding processes give results whose accuracy is as great as is permitted by the accuracy of the data, from 25% to 40% of the figuring is saved, and the false appearance of exactness is avoided."

THE CONSTITUTION OF ALGEBRAIC ABILITIES[1]

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The abilities acquired by a year's study of algebra may be constituted in very many different ways. A mathematician might think of the pupil as acquiring them by learning a few principles of notation, the laws of signs, the theory of exponents, the axioms, and the general rule that you can operate with literal numbers as with ordinary numbers. A mathematician who knew nothing of the conventional treatment of algebra in schools might begin to teach algebra as follows:

Let *a, b* and *c* represent any three numbers. Let $+, -, \times, \div$, the fraction line, $\sqrt{\ }$, $\sqrt[3]{\ }$, ^{2} and ^{3} be used as in arithmetic. Let ab or $a.b$ mean $a \times b$. Let ( ) mean that the expression within is to be treated as one number.

$+a = 0 + a \qquad +b = 0 + b \qquad +c = 0 + c$

$-a = 0 - a \qquad -b = 0 - b \qquad -c = 0 - c$

Adding $-a$ is the same as subtracting $+a$

Subtracting $-a$ is the same as adding $+a$

$(+a)(+b) = +ab$

$(+a)(-b) = -ab$

$(-a)(+b) = -ab$

$(-a)(-b) = +ab$

$$\frac{+a}{+a} = +1$$

$$\frac{+a}{-a} = -1$$

$$\frac{-a}{+a} = -1$$

$$\frac{-a}{-a} = +1$$

If equals are added to equals the results are equal.

If $a + b = c \qquad a + b + d = c + d$

If equals are subtracted from equals the results are equal.

If $a + b = c \qquad a + b - d = c - d$

and so on.

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Such a straightforward general treatment has a good deal in its favor, in the way of brevity and dignity. The experience of teaching, however, shows that algebraic abilities are not constituted in the minds of the pupils, out of a few general sweeping laws. No text-book or teacher of today, for example, would dare to assume that pupils who had been taught as above would be sure to understand that $(c - d) - (e + f)$ means "Subtract $e + f$ from $c - d$," or that

$$\frac{cd}{a} - \sqrt{a + b + c}$$

means "Subtract $\sqrt{a + b + c}$ from $cd \div a$," or even that $7cd^2 - 4cd^2$ means "Subtract $4cd^2$ from $7cd^2$."

The customary approved teaching of today builds up these abilities out of many detailed abilities. The pupil learns the meanings of about two hundred and fifty terms, such as:

Abscissa	Antilogarithm
Absolute term	Applying a formula
Absolute value	Ascending powers
Aggregation	Axes
Algebraic addition	Axiom
Algebraic expression	Base (distinct from power)
" number	Binomial
" product	Binomial theorem
" solution	Brace
Algebraic subtraction	

He learns about one hundred and fifty rules, such as:

Like roots of equals are equal.

To add a positive number to a negative number take the difference of their absolute values and prefix the sign of the numerically greater number.

$a - 0 = a$

$0 - a = -a$

To add similar monomials find the algebraic sum of the coefficients of the common factor and prefix this sum to the common factor.

He forms many habits, either as applications of these rules or as accessories acquired in the course of computation and problem solving. For example, the principle, "We can represent

numbers by letters " develops into at least ten distinct habits of thought, namely:

1. A letter may mean a particular number of things, like men, boys, or eggs.
2. A letter may mean a particular number of units, like cents, quarts, feet.
3. A letter may mean any one of a number of numbers, like the number of dollars in the cost of any number of suits of clothes of a certain sort, or the number of square feet in any rectangle.
4. A letter may mean any number, as in
$$(p + q)\ (p - q) = p^2 - q^2$$
5. If you call a certain number p, you may call 3 times that number q or r or s or any letter except p that you please, but it is commonly useful to call 3 times that number $3p$.
6. If you call a certain number p you may call 3 more than that number any letter except p that you please, but it is commonly useful to call it $p + 3$.
7, 8 and 9. The same principles of consistency and utility with $p + 3$, $p - 3$, $\frac{p}{3}$ and $\frac{3}{p}$
10. If we call a certain number (say, the profit Mr. A. made in Jan., 1922) p, we don't call it anything else and don't call p something different so long as we are thinking about that particular problem.

In spite of all our experience in teaching algebra we do not seem to have found the optimum constitution of algebraic abilities. Reformers like Rugg and Clark and Nunn not only eliminate certain abilities, change the emphasis on those which they retain, and add new ones; they also constitute the retained abilities in different ways. They would be the first to expect that other desirable changes will be found by further experimentation and improved insights.

To the psychologist who tries to follow through the mental operations of pupils from their first solutions of simple formulae to their comprehension of the parabolic relation, the binomial theorem, and the like, there seem to be many promising invitations to experiment, and even many cases where improvements

can be made at once by a straightforward application of the laws of learning. In the remainder of this article we shall note some of the general features of the constitution of these abilities, namely the provision for habits now neglected, the elimination of unnecessary habits, and the use of adaptable, even of loose, habits in place of inflexible rules which in actual learning have to be broken.

PROVISION FOR HABITS NOW NEGLECTED

The hundred and fifty rules do not cover all that the pupil must know and do. For example, throughout algebra the pupil has to decide when to indicate an operation only and when to carry it through according to some known routine. Suppose that he is faced with the need of dividing a by b, a^2 by a, a by a^2, and $\sqrt{625}$ by $\sqrt{10}$. In the first case he must only indicate $\frac{a}{b}$; in the second he must subtract with the exponents and write a; in the third case he must (according to usual methods of teaching) not subtract with the exponents, but indicate the division as $\frac{a}{a^2}$ and then divide each term by a; in the last case he is again customarily taught to indicate the division as $\frac{\sqrt{625}}{\sqrt{10}}$ and then proceed to $\sqrt{\frac{625}{10}}$ or $\sqrt{62.5}$, or to $\frac{25}{\sqrt{10}}$ or still worse to $\frac{\sqrt{625} \times \sqrt{10}}{10}$ or $\frac{\sqrt{6250}}{10}$. It would seem worth while to provide definitely for each useful habit, where there is some one procedure that is best, and to provide also a clear understanding that in certain cases you can do one of a number of things, any one of which is right so far as it goes, but some one of which will be best according to circumstances.

To take another illustration, in teaching division of a monomial by a monomial it is customary to use almost or quite exclusively tasks like $\frac{a^2}{a}$, $\frac{ab^2c^3}{ab^2c}$ and $\frac{27a^3b^3c}{9\,a^2b}$ where the quotient is integral—where no factor is left below the fraction line. $\frac{a}{a^2}$, $\frac{ab^2c}{ab^2c^3}$ and $\frac{9a^2b}{27a^3b^3c}$ commonly do not occur at all in the systematic treatment of

division, and occur very rarely in the reduction of fractions to simplest form. In a census of all the work, oral and written, computation and problem solving, provided for the first year's study of algebra in three standard instruments of instruction, we find the following enormous disparity between samples of the two types of task:

Let a, b, c, etc., represent numerals.

Let x, y, z, etc., represent literal numbers.

Then the averages for the three text-books are:

$$\frac{ax}{b} \text{ and } \frac{axy}{b} \text{ occur 848 times.}$$

$$\frac{a}{bx} \text{ and } \frac{a}{bxy} \text{ occur 2 times.}$$

$$\frac{ax}{x}, \frac{x^2}{x}, \text{ and } \frac{x^3}{x} \text{ occur 245 times.}$$

$$\frac{x}{ax}, \frac{x}{x^2}, \text{ and } \frac{x}{x^3} \text{ occur 8 times.}$$

This is an inadequate preparation for the actual use of algebra in later mathematical study, science, or the general work of life. The larger numeral, the persisting literal factor, and the higher power probably do go in the numerator oftener than in the denominator, but not in the ratio of 110 to 1. If fifteen-year-old boys and girls have a certain result happen 110 times as often as its opposite, they tend to think of that opposite as impossible or wrong. They will feel no surety when any division comes out with a balance in the denominator.

The customary procedure is due to the tradition that considered fractions as a difficult matter, not to be touched in any way until all operations with integers had been mastered, and to the effort to fit the pupils' operations to the two rules of dividing by subtracting exponents and of reducing fractions by dividing both numerator and denominator by the same factor. The tradition is simply nonsense, a fallacious notion due to analogy with arithmetic and the idea that because fractions as a whole are hard, everything about them is hard. The fraction form is wisely used in the division of a monomial by a monomial. The plan of fitting the pupils' operations to the two rules has much in its favor and can reasonably be retained, the defect in the

pupils' habits being remedied by extending the notion of division when reduction of fractions is learned, and by giving suitable practice then and thereafter. It would, however, perhaps be still more effective to make the procedure general from the beginning by introducing —1, —2, —3 and 0 as exponents, teaching their meanings, and permitting either a^{-1} or $\frac{1}{a}$ as an answer to $\frac{a}{a^2}$ or $\frac{a^2}{a^3}$, and similarly for all factors remaining below the fraction line. Teachers in general will be shocked at the proposal to teach anything about negative exponents at this stage, but the most acute teachers may not be. For some of them have perceived by intuition and experience, what psychologists infer from general principles, that it is often very advantageous to get used to a few elements of a topic or doctrine long before the doctrine as a whole is systematically taught. Thus in arithmetic we now teach certain facts about the addition of fractions two or three years before the general treatment of the addition of fractions, and teach addition, subtraction, multiplication and division with United States money long before the general treatment of decimals, and do so with great profit, both to the early work and to the general treatment when it comes.

THE ELIMINATION OF UNNECESSARY HABITS

We teachers and learned people are obsessed by the tendency to learn everything about anything and teach everything about anything that attracts our intellects. The former tendency is, in spite of certain pedantries, perhaps the greatest blessing the world has. The latter is also a blessing on the whole, but it can be made a greater blessing if we control it. It has needed control in algebra. The educational reformers have had to work hard to convince teachers of mathematics that it is not wise or humane or scientific to burden the learning of algebra by children with all the factorizations ingenuity can devise, or to hide $x^2 - y^2$ under all the disguises through which mathematical acuity can penetrate.

Consider the case of the teaching of radicals as a sample. Suppose that we eliminated all the customary work with radi-

cals and the customary general systematic treatment of powers during the first year, and instead taught the facts now taught much later concerning fractional exponents and negative exponents, plus the following:

1. $\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$; if $a = b$, $a^m = b^m$,—the rule for principal roots.

2. $\sqrt{a}$ means $a^{\frac{1}{2}}$ $\quad$ $\sqrt[3]{a}$ means $a^{\frac{1}{3}}$ $\quad$ $\sqrt[4]{a}$ means $a^{\frac{1}{4}}$, etc.
$\sqrt{a^3}$ means $a^{\frac{3}{2}}$ $\quad$ $\sqrt{a^4}$ means $a^{\frac{4}{2}}$ $\quad$ $\sqrt{a^5}$ means $a^{\frac{5}{2}}$, etc.
$\sqrt[3]{a^2}$ means $a^{\frac{2}{3}}$ $\quad$ $\sqrt[3]{a^3}$ means $a^{\frac{3}{3}}$ $\quad$ $\sqrt[3]{a^4}$ means $a^{\frac{4}{3}}$, etc.

3. If in any task you meet an expression with $\sqrt{\ }$ or $\sqrt[2]{\ }$ or $\sqrt[3]{\ }$, change it to an equivalent expression with the proper exponents (and with parentheses if necessary).

4. If, after multiplying, dividing, raising to powers and finding roots by proper use of the exponents, any arithmetical number still has an exponent other than 1, evaluate by using tables of powers and roots, or by logarithms, or by trial and correction if you cannot get the necessary tables.

5. If, however, you see some way of shortening the work, as by $(5^{\frac{1}{2}} + 7^{\frac{1}{2}})(5^{\frac{1}{2}} - 7^{\frac{1}{2}}) = 5 - 7$, take it.

This suggestion will, like the one on page 410 shock many teachers. They will think, or rather feel, that the general procedure with exponents is too hard to teach thus early, that $\sqrt{\ }$ should precede, not follow $^{\frac{1}{2}}$, and that it will seriously mutilate algebra to omit such stimuli to ingenuity as $\sqrt[3]{9}(27)\sqrt{3}$, or $(\sqrt{8} + \sqrt{7})(8 + 2\sqrt{7})$, or $\sqrt[4]{\frac{2}{3}}$, or $\frac{2 - \sqrt{5}}{3 + \sqrt{2}}$ or to replace them by equivalents with fractional indices whose simplifications are aided by tables.

These objections are instructive. The difficulty pupils have in learning the general procedure with fractional and negative exponents is largely of our own manufacture. In the ordinary course of events what they learn about roots, radical signs, surds, and rationalizing hinders them in learning the general treatment. "If it is 2 in $\sqrt{\ }$ how can it be $\frac{1}{2}$? If it requires two distinct operations in $\sqrt[3]{8^2}$ how can you express it as a single fraction $8^{\frac{2}{3}}$? And why does it turn bottom side up? I was taught to turn $\sqrt[3]{a^4}$ into two a's $(a\sqrt[3]{a})$ as soon as I saw it. Is it reasonable for

me to call it $a^{\frac{1}{3}}$ now? In any case why not leave me to do it in peace in the old way that I learned at such cost?'' Such is the unconscious argument of the pupil's nervous system. In the ordinary type of 'explanation' of the procedure with exponents, we are prone to add to its difficulties for all save the gifted pupils. We explain how it can be true, justify it, and present it in the highly abstract rules

$a^m \times a^n = a^{m+n}$ whatever m and n may be
$a^m \div a^n = a^{m-n}$ whatever m and n may be
$(a^m)^n = a^{mn}$ whatever m and n may be

and quiz the pupils lest they be not convinced.

Now the difficulty is not that the pupils feel logical objections, or any intellectual shock at the innovation. Unfortunately many of them would not rebel intellectually if they were told that hereafter a^2 would always mean $10a$, a^3 would mean $100a$, and so on. Nor are they helped greatly to understand the reasonableness of the new system by $a^m \times a^n = a^{m+n}$, etc. Their chief difficulty is that they are not *used to it* and do not realize *what* they are doing. They feel lost and dizzy with these strange exponents. They need first just to become at home with $(a^{\frac{1}{2}} + b^{\frac{1}{2}})$ $(a^{\frac{1}{2}} - b^{\frac{1}{2}})$ and the like. They need, second, to see how the results do turn out by this new procedure, to see that $a^{\frac{5}{2}}$ means $a^{2\frac{1}{2}}$ and is about half way between a^2 and a^3, that 10^{-4} is $\frac{1}{10000}$, and the like. Such series as the following are useful for this:

$10^4 = 10000$	$10^{\frac{1}{2}} = 3.16$	$9^{\frac{8}{3}} = 350 +$
$10^3 = 1000$	$10^{\frac{1}{3}} = 2.15$	$9^{\frac{7}{3}} = 168 +$
$10^{2\frac{1}{2}} = 316 +$	$10^{\frac{1}{4}} = 1.78$	$9^{\frac{6}{3}} = 81$
$10^2 = 100$	$4^3 = 64$	$9^{\frac{5}{3}} = 38.9$
$10^{1\frac{3}{4}} = 56.2$	$4^{5} = 32$	$9^{\frac{4}{3}} = 18.7$
$10^{1\frac{1}{2}} = 31.6$	$4^{\frac{4}{2}} = 16$	$9^{\frac{3}{3}} = 9$
$10^{1\frac{1}{4}} = 17.8$	$4^{\frac{3}{2}} = 8$	$9^{\frac{2}{3}} = 4.32$
$10^1 = 10$	$4^{\frac{2}{2}} = 4$	$9^{\frac{1}{3}} = 2.08$
$10^{\frac{3}{4}} = 5.62$	$4^{\frac{1}{2}} = 2$	
$10^{\frac{2}{3}} = 4.64$	$9^3 = 729$	

They need, in the third place, to have enough practice in the operations so that they can do them correctly and have their

minds free to consider what they are doing, and what comes of it, and why it is a useful thing to do, and how the formulas, $a^m \times a^n = a^{m+n}$ and the rest, sum up in a beautiful set of rules a great many operations which they have learned to perform and to trust as sure means of obtaining correct results. The very pupils to whom these formulas are baffling verbal edicts at the beginning of the learning may, after enough instructive practice and checking, find them admirable summaries of what they have learned.

The second objection, to the effect that $\sqrt{\ }$ and $\sqrt[3]{\ }$ and $\sqrt[4]{\ }$ should precede $\frac{1}{2}$ and $\frac{1}{3}$ and $\frac{1}{4}$, can hardly have any other origin than mistaking what is familiar to us as what is easy and natural[1]. $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ are better symbols than $\sqrt{\ }$ $\sqrt[3]{\ }$ $\sqrt[4]{\ }$ in every way whatsoever save two, for all pupils whatsoever. Their two demerits are, of course, the likelihood of confusing them with common fractions, and the more common use of $\sqrt{\ }$ $\sqrt[3]{\ }$ $\sqrt[4]{\ }$.

Pupils in high school are protected against the confusion with fractions by long habituation to a^2, a^3, a^4, etc. They should learn to understand both types of symbol, but $\frac{1}{2}$, $\frac{1}{3}$, etc., should precede. These should indeed be used in the first treatment of roots in algebra, whenever that may be, the pupil being taught that the square root of a in algebra is written $a^{\frac{1}{2}}$, or, sometimes, $\sqrt{a}$.

The third objection was that the rule permitting students to get rid of certain surds by direct use of tables, logarithms or computation instead of searching for clever ways to rationalize would rob algebra of a certain portion of its intellectuality. It would, but the general addition of intellectuality by making the treatment of radicals as a whole a logical and straightforward application of the theory of exponents, far outweighs the loss. Radicals as a whole as now taught are a confused set of maxims plus a bag of clever tricks.

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[1] It seems probable that the antipathy to fractional exponents, founded on their unfamiliarity, is made much worse by the strain that commonly attends reading them because the type used is so small. The traditions of the printer in this respect are very bad, and the authors of text books have weakly given way to him. Fractional exponents should, of course, be printed in type that is easily legible.

On the whole it seems conservative to estimate that at least three fourths of the time now given to radicals could be saved with no loss, but a net gain to the pupil's development.[1]

ADAPTABLE HABITS VERSUS INFLEXIBLE RULES

The rigor of its definitions and the universality of its rules are among algebra's chief merits, and we should cherish them. We do not, however, dignify algebra by claiming a universality when it is really lacking, nor by asserting something rigorously and evading it soon thereafter. When the definition works only part of the time and the rule is helpful for only some of the cases, it may be better to organize the learning as a set of rather loose and opportunistic habits than to insist on the rule and then qualify and amend and even break it.

Consider these two definitions and these two rules:

(1) If an expression is separated into two factors, either factor is called the coefficient of the other.

(2) Terms which differ in no respect or only in their coefficients are called like terms.

(3) To add like terms add the coefficients of the common factor and prefix this sum to the common factor.

(4) To add polynomials we write like terms in the same column and add these terms, writing their sums as a polynomial.

Consider $4ab^2c$. By the definition (1), b^2 is called the coefficient of $4ac$, and it would be reasonable to call it so, but in actual fact it isn't so called. $4ab^2$ by the definition is the coefficient of c.

Consider $5a^2b^3c$, regarding one factor as $5a^2b^3$ and the other as c. By the definition (2), $4ab^2c$ and $5a^2b^3c$ are called like terms, and it would be reasonable to call them so, but teachers would not call them so in one case out of fifty, and might rebuke pupils if they so called them.

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[1] It may be noted that an appreciation of the desirability of using the general procedure with exponents in the computations with radicals is shown by certain teachers and authors of textbooks who insert more or less of the general treatment along with the older teachings as an *alternative* way of handling the computations. This, however, seems to be of little or no avail in lightening the pupil's load. We do not wish to have him learn the inferior way and that there is a better way, but to be blissfully ignorant of the inferior way. We wish him only to turn $\sqrt{\ }$ into exponential form and to know the common equivalents so that he can read books using $\sqrt{\ }$. Apart from that, the less he does with $\sqrt[3]{a^2}$, etc., and the more exclusively he works with a[illegible] a[illegible], a[illegible], etc., the better pleased we should be.

Consider:—

Add $3abc + 4ab^2c + 5a^2b^2c$ to $6abc + \frac{16abc^2}{c} + 7cef + 8abc$.

Applying rules (3) and (4), which shall we put in the same column, all of the terms or only the first and last, or the first, last and second from the last?

In fact it is not the definitions and rules, but the habits developed by experience with particular numbers and expressions that teach a pupil what to regard as the coefficients, and what terms to collect together in addition. A practical joker could arrange a set of exercises in early addition that would force a conscientious pupil who tried to follow the definitions and rules into utter bewilderment. Would it not be better to build up useful habits of addition and subtraction without the pretence that they are logical deductions from the definitions and rules?

"An algebraic expression in which the parts are not separated by + or — is called a monomial" is given as a rigorous definition; but later the pupil has to consider $3(ab - cd)$ as a monomial; and later he must treat $\frac{12a^2b^2c}{3\,abc}$ as two monomials though there is no separation by + or —.

A special case of importance is the erection of technically convenient procedures like arranging in ascending or descending powers before multiplying or dividing, writing like terms under one another, or beginning a monomial with its numerical factor, into fixed rules on a level with the most general axioms and principles. Every such rule which a pupil finds that he can break (as he can all such) without getting wrong answers means a risk that he will lose respect for and confidence in the really imperative rules. If we leave to habit everything that can be done as well by habit, we gain an added dignity for the matters that really are matters of principle. For example, it seems sound psychology to teach "If equals are added to equals the results are equal" as a rule, but one should teach "transposing," universally valid though it is, as a convenient habit. It is not because he does not value rules and principles in algebra that the psychologist often prefers to use habits instead; it is rather because he does value principles and does not wish them to be misused and cheapened.

ROMANCE IN SCIENCE

An Experimental Course Offered by a Department of Mathematics[1]

By PROFESSOR BESSIE IRVING MILLER
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At the beginning of the second semester of this year we opened a two-hour course, popularly called "Browse," with a description of a sleek, cream-colored Jersey, standing in deep grass and pausing, after a lucious bite, to gaze meditatively over the landscape. Before the spring vacation, in that same course, I gave an examination on the content of the Einstein Theory and its relation to the Newtonian Theory. Why, how, and with what results I made that transition are the subjects of this paper.

Two years ago at a section meeting of the Association, Dr. Slaught urged that, since we mathematicians had an interesting subject, we might profitably take time occasionally to show the students how interesting it was. This, combined with a previously observed fact that the survey course of English literature not infrequently left the science student with either a distaste for reading of any sort, or with the expectation of usually being bored by it, galvanized me into making some more definite observations, with the purpose of getting a solution for the pedagogical problem presented.

The results of these observations on my own classes are as follows: first, that a liking for reading can be aroused in the scientific student by references to scientific literature, not to text-books; second, that interest in scientific ideas can be aroused in the arts student by relating mathematics to other fields of knowledge through essays; third, that a greater lack of scientific literary background than was usual in the preceding generation exists now among all types of students.

With these results in mind I announced the new course under the double title "The Romance of Science or The Grammar of Science, Whichever You Wish," to show the arts student that science had the imaginative quality which she so enjoyed, and to show the science student that poetry, fiction and opinions did not constitute the whole of the world's literature.

[1] This paper was read before the meeting of the Illinois section of the Mathematical Association, April 28, 1922.

In the college catalogue the course is described as being " Some Literature of Science," a somewhat commonplace title, which would not attract too much attention on the part of the ultra-conservatives in scientific teaching. " Browse " is the unconventional nickname which was given the course because of its appropriateness to the purpose without having the unfortunate connotation for the student that words such as " Meditate " might have, for no American student is willing to acknowledge that she thinks, much less meditates.

The prerequisite for the course is Freshman Mathematics or enrollment in that course with a grade of " B " for the first semester. This secures some knowledge of analytical geometry, limits and differentiation on the part of all, although it is a very elementary knowledge.

The main topics of the course are six in number. The first was the scientific method, ancient and modern. This was illustrated by the various theories of the earth's motions, which led to Newton's Law of Gravitation, and which included a study of hypothesis and conclusion. The second was dimensionality in algebra and geometry, including, of course, the fourth dimension; and the third, the distinction between Euclidean and Non-Euclidean geometries. These two involved a discussion of some popular fallacies and also of the scientist's point of view, when he chooses the most convenient mathematical hypothesis from several, any one of which is equally logical. The fourth topic was the Einstein Theory, which involved no proof, but included a description of the theory and of its relation to the Newtonian Theory, a statement of the mathematical and experimental verifications up to date, and the present status of the theory in the scientific field. The fifth topic, which I am now working on, is applications of mathematics in other fields than that of physics. This includes a discussion of the human value of mathematical thinking and the place held by mathematics in civilization. The last subject will be some of the famous concepts of mathematics, such as number, limit, function, group.

In addition to the main fields of investigation indicated above, side excursions into the domains of art, astronomy, biology, chemistry, mathematics, medicine and physics are made. This is

possible because of the method of conducting the course. Once a week references and lectures are given on one of the main topics. Supplementary reading of two kinds are assigned over a considerable period of time. Each student chooses one long or two short books for her " browsing " throughout the semester. She is guided by only her own taste except that her selection must be made from the official bibliography of the course. In addition, essays or short books are so assigned that each student will have read by the end of the semester one or more pieces of good literature in each of five scientific fields not covered by her " browse " book. A second class appointment is devoted to discussions or speeches or papers. The last two are sometimes expository and sometimes critical.

Some of the more obvious results of the course can be listed as follows:

1. An answer to the student's query, " Why study mathematics? "

2. An increased vocabulary for all and a remarkable improvement in correctness of speech and flow of language for the science student.

3. A rebutal of the statements, " Books Are Dull," and " Science Is Dull," made by the science and arts students respectively.

4. An extension of the intellectual horizon and a consequent increase in intellectual tolerance.

5. A knowledge of a scientific but literary reading list.

6. A knowledge of some fundamental concepts of mathematics, the lack of which Henry Adams so bewails.

7. A knowledge of some of the modern developments of mathematics. After further experimentation it is probable that a coefficient can be attached to some of these results.

The less obvious results aimed at by the course, which I cannot now measure and perhaps may never be able to measure, are a permanent taste for reading and even more for " browsing." The necessity of " browsing " is rather well expressed by Arnold Bennett in the last chapter of his little book called *Literary Taste, How to Form It*. I shall quote a little from this chapter called "Mental Stocktaking" in the hope that you will be

lured on to read more. "The superlative cause of disastrous stocktaking remains. * * * It consists in the absence of meditation. * * * if a man does not spend at least as much time in actively and definitely thinking about what he has read as he has spent in reading, he is simply insulting his author. * * * reading with him is a pleasant pastime and nothing else. This is a distressing fact. * * * It is distressing for the reason that meditation is not a popular exercise. If a friend asks you what you did last night, you may answer, "I was reading," and he will be impressed and you will be proud. But if you answer, "I was meditating," he will have a tendency to smile and you will have a tendency to blush. I know this. * * * (I cannot offer any explanation). But it does not shake my conviction that the absence of meditation is the main origin of disappointing stocktakings." Most of us who teach will agree with Mr. Bennett.

Although the results from the course have been very satisfactory, there are some difficulties and dangers inherent in the very character of the course. Depth may be sacrificed to breadth. Confusion may be caused by the introduction of too many ideas, in which event it would have one of the disadvantages of the literature course whose effect upon science students is so often to be depreciated. Idle dreaming may be cultivated instead of leisurely thinking. But it is not very difficult to avoid these dangers, and they are offset somewhat by the flexibility of the course and its very general appeal, for it can be given each year with variations, can be adapted to all classes of students by means of the supplementary reading, can be used to eliminate popular fallacies concerning the study, content and applications of mathematics, and to stimulate interest in all the sciences. It has certainly proved successful if the students' interest, the extraordinary improvement shown by some, and the very high quality of work attained by others can be used as criteria.

The reading list which follows contains not necessarily the

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[1] Many have asked if it is possible to work up a "Browse" course while it is being given. It seems to me to be impossible if the lectures are to be well done. A year's reading and listening to many lectures were needed to produce the two lectures on the Einstein Theory which were given so as to convey a real meaning to the class with only the preparations mentioned in the paper. A superficial knowledge of any of the subjects presented in the courses would seem to me to be fatal to the scholarship of the course.

—B. I. M.

best books which could be found anywhere, but the best which were accessible in our library this semester. Moreover, it does not aim to be exhaustive, but suggestive.

READING LIST FOR "BROWSING"

In the Spring of 1922

Mathematics XVIII (speaking technically)

Astronomy (A)

1. *Astronomy.* Hinks. This is an elementary, nontechnical little book, written in a somewhat commonplace style and addressed to any man.
2. *Origin of the Earth.* Chamberlain. This is well written, presents interesting facts and theories, is trustworthy scientifically and is addressed to an audience of intelligence but having little technical training. This was apparently the source of Prof. Salisbury's lecture last year.
3. *The Sun.* Sampson. This is one of the set of small, scientific books, written in a good and interesting style, for popular reading, where popular is used in the best sense of the term. They require thinking, but not too much; they are printed very clearly and are easy to hold. In fact their advantages are many, but the shade of their binding, an unbelieveable shade of pink, is somewhat wearing.
4. *The Earth.* Poynting. This is another of the pink series.

Art and Music (Am.)

1. *Curves of Life.* Cook. This is a large, well illustrated book giving illustrations of some mathematical curves, drawn from the field of biology, art, etc. Much of it can be read by freshmen.
2. *Physical Basis of Music.* Wood. This is one of the best of the pink series. There is little in it that is not very enjoyable.

Chemistry (C)

1. *Creative Chemistry.* Slosson. This is so popularly written that it gives incorrect scientific ideas by the assignment of the powers to feel and to will to inanimate objects. The Egyptians of the Pharohs' times would probably have approved of it. Its facts however are trustworthy.

Biology (B)

1. *Descent of Man.* Darwin. Darwin needs no introduction. You might read it and find that he did not say what he is frequently said to have said.
2. *Origin of Species.* Darwin. This proved to be very delightful reading one summer, spent in the mountains. It served as a transition from the writing of a mathematical dissertation to a "Sherlock Holmes" story.
3. *Living Plants.* Ganong. Such an artistic piece of scientific writing is rarely seen in these days of industrial science, except in mathematics. Ganong's book is as scholarly as it is interesting, as clear in its style as in its type.
4. *Life of the Spider.* Fabre. I have not read it but know it by reputation.
5. *Heredity.* Doncaster. This belongs to the pink series.
6. *Coming of Evolution.* Judd. This belongs to the pink series.
7. *Harvard Classics No. 38.* Harvey, Jenner, Pasteur, etc.

D

(This includes those books which are different or impossible to classify.)

1. *Harvard Classics No. 30.* Physics, chemistry, astronomy, geology. Faraday, Helmholtz, Kelvin, Newcomb, Geikie.

2. *Harvard Classics No. 33.* Physiology, medicine, surgery, geology. Harvey, Jenner, Lister, Pasteur.
3. *Harvard Classics No. 29.* The Voyage of the Beagle. Darwin.
4. *Discourses. Biological and Geological Essays.* Huxley.
5. *The Forest.* S. E. White. This is not scientific but it is a well written description.
6. *Jungle Peace.* Beebe. This is a charming piece of writing in the familiar essay style, and about such interesting things, that one lingers in reading in order to prolong the pleasure.
7. *Science and the Nation.*
9. *Realities of Modern Science.* Mills. The style is clear, but commonplace. The facts presented are interesting and readily understood. The book gives one quite a good idea of the scientific method.
10. *Outlook for Research and Invention.* Hopkins.
11. *Grammar of Science.* Pearson. The first part is interesting, the middle rather dull, the last part somewhat interesting.
12. *Education of Henry Adams.*

GEOLOGY (G)

1. *Earth Features.* Hobbs. This is addressed to a non-specialized intelligent reader. The illustrations are drawn from the United States, especially the north and west.
2. *Physiography.* Salisbury. This is long but interesting.
3. *Geology.* Lyell. This is old but interesting.

MATHEMATICS (M)

1. *Flatland.* A Square. This small book is useful rather than beautiful, but it is amusing and stimulating, if read with care at the beginning, and glanced at casually towards the end. It is easy to understand and yet gives the idea of dimension in space so that it can be applied.
2. *John Napier and the Invention of Logarithms.* Hobson. This is a half hour's reading about an instrument all Rockford graduates have used.
3. *Number System in Algebra.* Fine. This is interestingly written and not hard to read. It deals with that familiar but little understood thing called numbers and tells how they came to be.
4. *Study and Difficulties of Mathematics.* De Morgan. This is pleasant reading. It is one of the books bound in red.
5. *Mathematical Recreations.* Ball. This is full of games and puzzles as well as of other things. It is something of a scrap book.
6. *Mathematical Essays.* Schubert. If you enjoy the somewhat serious essay as a form of literature, you will probably enjoy some of these.
7. *Memorabilia Mathematica.* Mortiz. If you want an epigram or a definition, a topic for a paper or a proof of the diversities of men's ideas, read a few bits here and there. The book is a collection of notable sayings and writings either by or about either Mathematics or mathematicians.
8. *Human Worth of Rigorous Thinking.* Keyser. This is a collection of essays and lectures. Numbers I-VIII, XIV, XV are of general cultural value. The style is excellent and the ideas are sometimes interesting and sometimes inspiring.
9. *Fundamental Concepts of Algebra and Geometry.* Young. This is non-technical in that it assumes no knowledge of mathematics beyond elementary high school algebra and geometry. It is a series of lectures addressed to an audience interested in mathematics.
10. *Foundations of Geometry.* Hilbert. This is perhaps the most polished expression and the most scholarly treatment of the subject that has ever been written.

11. *Fundamental Concepts of Modern Mathematics.* Richardson and Landis. This is one of the disappointments of this generation. Its mathematics cannot be trusted.
12. *Philosophy of Mathematics.* Shaw.
13. *Space and Time.* Schick. This gives Einstein's theory, omitting the mathematical proofs.
14. *Organization of Thought.* Whitehead. This is more interesting than 15 both in style and content. Chapters I, II, VI, VII, VIII are perhaps the best. I read them with pleasure when sitting in the lobby of the Auditorium Hotel, where one may while away many an hour, musing over a book of this type and collecting data on human nature. It is not a book to hurry through, yet it is not so profound as to require one's study for its perusal. Perhaps this is because it consists of a series of lectures originally addressed to an attentive audience but one which was enjoying a vacation from routine duties. It is non-technical.
15. *Principles of Natural Knowledge.* Whitehead. This is an "enquiry". It has a most interesting origin you will find, if you read the introduction. The book is concerned with theories of the nature of matter, and the ultimate data of matter, the relation of geometry to science, the effects of the theories of relativity. If you think you know all about lines and points, read and be disillusioned.
16. *Foundations of Science.* Poincare. This requires maturity of mind, not a specialist's training. Remember it, so that when you reach the point where much that you read seems commonplace, because of its familiarity, and you really have the thrill of coming in contact with a new idea or a new correlation of ideas, where the beauty of the abstract seems greater than the beauty of the concrete, then read Poincare, leisurely, sometimes dreaming, sometimes meditating, sometimes experimenting, but always thinking. Then the simplicity of the language, the dignity of the concepts, the inspirational suggestiveness of the thought may give you that permanent acquisition, a moment in which you live nobly and below which you may never sink as low as you have in the past. But beware, lest, having eyes, ye see not.
17. Science. March, 1912. *Organization of Knowledge.*

Physics (P)

1. *Wireless Telegraphy.* Fortescue.
2. *The Atmosphere.* Berry
3. *Beyond the Atom.* Cox. These three belong to the pink series.
4. *Experimental Researches in Electricity.* Faraday. Notice the motto used as the frontispiece.
5. *Soap Bubbles.* Boys.
6. *Light, Visible and Invisible.* Thompson. This has a pleasing style, interesting contents and is easily understood.
7. *Surface Tension.* Willows and Hotschek.
8. *Electricity and Matter.* Thompson.
9. *The Electron.* Milliken.
10. *Sound.* Tyndall.
11. *Light.* Tyndall.
12. *The New Physics.* Poincare.

SOME MATHEMATICS OF THE CALCULATING MACHINE

By L. LELAND LOCKE
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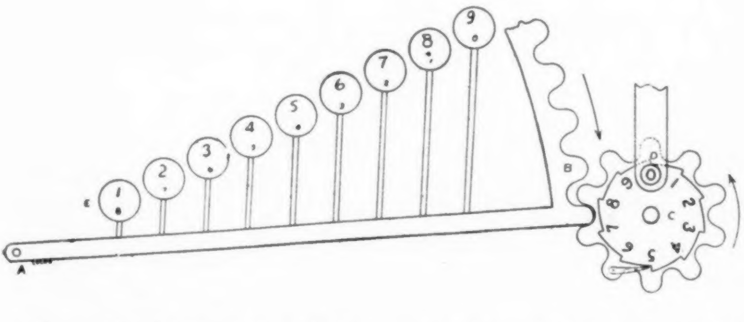
There are certain relatively simple properties of numbers which would be deemed of little importance in the science of numbers except for their application in the field of machine calculation. In considering such properties no detailed knowledge of the mechanical features of the machine is necessary. We are concerned only with the *set-up* and final *recording* or *registering dials.* One unit of our primitive machine will consist of a set of nine keys numbered from 1 to 9, and a registering dial actuated by the keys in such a manner as to turn one tenth of a revolution for each unit in the number on the key pressed down. In the diagram, if O appears at window D, over registering dial C, and if key 7 is pushed down, the segmental gear B causes dial C to rotate in a counter clockwise direction until 7 appears at the window. If key 5 is then operated the dial C is turned 5 more spaces and 2 appears at the window. This primary actuation will be called *digital addition,* as only the unit's digit of the sum is registered.



If, as the 9 leaves the window, means is provided to move the dial in the next, or ten's place, forward one step, i. e., to provide for the *carry,* the primitive machine is complete. Neglecting the mechanical features necessary to make it operable, we have essentially an *adding* machine. It is the purpose to set forth what may be done with such a machine and why.

Multiplication is performed by repeating the addition. To multiply 849 by 37, use the 9 key in the first column, 4 key in the ten's column or second column, and the 8 key in the third or hundred's column. Obviously the operation of this set of keys simultaneously 37 times would accomplish the multiplication. The work in such a case would be prohibitive. After the set has been operated 7 times each finger is shifted one place to the left, placing the finger on the 9 in ten's column, etc., and operated 3 times, i. e., multiplying by 3 and by 10 at the same time.

Subtraction is an inverse process. To make the machine process an exact parallel of the arithmetical, the registering dials must rotate in the opposite direction when subtracting. With an adding machine, as described above, the registering dials rotate in but one direction. Subtraction on such a machine is performed by what may be called *over-addition,* in the explanation of which process several terms will be needed.

The *complement of a number* is the difference between that number and the next higher power of ten. The complement of 6 is 4.

The *co-digit* of a digit is the difference between that digit and 9. The co-digit of 6 is 3.

The *supplement*[1] *of a number* will indicate the difference between that number and an infinitely great power of ten. The supplement of 257 is . . .999999643.

To write the supplement of a number, write the complement of the right hand significant figure, proceed to the left with the co-digits in order and complete the capacity of the machine with 9s. Some machines are equipped with a device to cut out the carry mechanism at any desired point. This device renders the prefixed 9s unnecessary.

Examples

1.	2.	3.	4.	5.
7351	7351	7351	7351	7351
—1867	+8133	+99998133	—1800	+8200
5484	1/5484	00005484	5551	1/5551

[1] The writer suggests this word to signify a concept for which no term has been adopted. The 'infinite' phase of the definition amounts in practice to prefixing sufficient 9's to prolong the carry beyond the capacity of the machine.

Examples 1 and 4 indicate arithmetical subtraction. In 4, the 8 is the right hand significant figure, the complement of which is 2 in example 5.

Examples 2, 3 and 5 indicate machine subtraction. In 3 the supplement of the subtrahend is used; in 2 and 5 the carry cut-off is indicated by the line, the 1 at its left not appearing on the machine.

To simplify the picking out of the co-digits in subtraction, the co-digit appears in small or in red type on the digit key. Since the complement of the right hand significant figure of the subtrahend is used and not the co-digit, the operator must move the finger up one key for this particular order.

The algebraic equivalence of the results is seen in the case of a three order number as follows:

Subtraction

$$\begin{array}{lll} 100c & +10b & +a \\ 100z & +10y & +x \\ \hline 100(c-z) & +10(b-y) & +(a-x) \end{array}$$

Adding the Supplement to the Subtrahend

$$\begin{array}{lll} 100c & +10b & +a \\ +100(9-z) & +10(9-y) & +(10-x) \\ \hline \end{array}$$

$$900 + 100(c-z) + 90 + 10(b-y) + 10 + (a-x) = \\ 1000 + /100(c\text{-}z) + 10(b\text{-}y) + (a\text{-}x)$$

Division. Division may be performed on an adding machine by repeating the subtraction, shifting the subtrahend to the right when the partial dividend becomes smaller than the divisor. The subtraction is performed by overaddition and the quotient figures are written as found. The process may be described as follows:

Set up the dividend.

Depress the keys indicating the supplement of the number as far to the left as possible, using the co-digit keys as in subtraction. Repeat the depression of this set of keys, counting the depressions, until the partial dividend is less than the divisor. Shift the fingers each one place to the right and continue.

It will be evident that such a process requires either a very intimate knowledge of the science of numbers or sufficient train-

ing to make the process mechanical, to give the operator that sense of confidence in the work which is essential.

Division on an adding machine is simplified somewhat by using the complement of the subtrahend rather than the supplement, by virtue of the following unique property of numbers. It will be noted that while the use of the supplement cuts out the carry mechanism immediately at the left of the number, the use of the complement does not.

If the complement of the divisor is repeatedly added to the partial dividend until the number of times the addition takes place is equal to the digit on the left of the partial dividend, including the accretions from the carrys, the partial dividend at that point is the actual dividend remaining in that division which produces a quotient figure equal to the number of additions at this point. An example will make this clear. Let it be required to divide 258 by 37. 63 is the complement of 37.

Machine Work

	No. of additions
2 | 58	
63	
3 | 21	1
63	
3 | 84	2
63	
4 | 47	3
63	
5 | 10	4
63	
5 | 73	5

By Division

37)258(5
185
———
73

In this example, when 5 additions have been made, the right hand figure of the sum is 5. At this point the right hand part of the sum is the partial dividend 73 when the quotient figure is 5. If this partial dividend is still greater than the divisor, continue the process in this order. In the example given, the 73 is the new true partial dividend and the process of division begins anew at this point, giving, in this case, one more unit in the present order of the quotient.

The significance of the above property of numbers will be lost unless it be kept in mind that every division is made up of a series of division examples, each example having its own partial dividend formed when the next figure is brought down and the division beginning anew at this point. With the machine process each new division begins at that point when the true partial dividend appears on the machine. A little reflection will show that the separation into simple divisions in the machine process does not always coincide with that of the arithmetical process.

The following algebraic treatment of the property is readily seen to be general.

In dividing $100a + 10b + c$ by $10d + e$ let $a < d$ and let the quotient be $a + n$ and the remainder be $10g + h$.

$$10d + e\ )\ 100a + 10b + c\ (\ a + n$$
$$10g + h \text{ remainder}$$

The a is the first digit in the dividend and the n is the accretions from the carrys. To prove that the addition of the complement of the divisor $(a + n)$ times produces the correct partial dividend corresponding to the subtraction of the divisor $(a + n)$ times.

$$\text{The complement of } 10d + e \text{ is } 100 - (10d + e \text{ or}$$
$$10(9 - d) + 10 - e$$
$$(a + n)\ [100 - (10d + e)] =$$
$$100\,(a + n) + 100a + 10b + c - 10ad - 10nd - ae - ne$$

It is necessary to show that this expression lacking the first member is equal to the remainder $10g + h$.

$$(10d + e)\ (a + n) + 10g + h = 100a + 10b + c$$
$$\text{or, } 10g + h = 100a + 10b + c - 10ad - 10nd - ae - ne$$

As each step of the division is performed the quotient figure appears in its proper place following the first one, until the division is completed. The final appearance of the machine is quotient | remainder, the $a + n$ being the quotient.

If the complement of the divisor is made up of many orders or if the proper keys are too inconveniently placed to be easy of manipulation, the above process is apt to be unwieldy. Short division by this process is tedious and for it is substituted multiplication by the reciprocal of the divisor, a table of reciprocals being provided. Square root by subtraction of successive odd numbers and the principles underlying automatic division will be considered in another article.

"STERADIANS" AND SPHERICAL EXCESS

By PROFESSOR GEORGE W. EVANS
Charleston High School, Boston.

In elementary geometry, or oftener in trigonometry, we speak of radian measure of plane angles; but, if we ever mention the measure of a solid angle by the included area of a unit sphere, it is a mere comment, and seems to have nothing to do with the fact that the area of a spherical triangle is proportional to its spherical excess. It is not easy for the pupil to infer, and he generally does not infer, that the spherical excess, expressed in radians, is precisely this measure of the solid angle, and, if multiplied by r^2, gives the area of the triangle.

Quite as mysterious is the way in which the perimenter of the polar triangle appears opportunely to help us out with the spherical excess, coming from a dimly remembered time when we were talking about the face angles of a polyedral, and vanishing again into a useless attic.

What I am now suggesting is a method of connecting up these loose ends, and of extending the results to the areas of convex polygons and circles on a sphere, and to the measure, in "steradians," of convex polyedral angles and right circular cones. A steradian is defined as the solid angle which, having its vertex at the center of a sphere of radius r, intercepts the area r^2 on the surface of that sphere.

Polar Polygons

In any sphere call the center V and any spherical polygon $ABCDE$ . . . : then the polar polygon $A'B'C'D'E'$ . . . . is determined by drawing radii VA' perpendicular to VBC, VB' to VCD, and so on. If these perpendiculars are all drawn outwards with respect to the polyedral angle V-$ABCDE$ . . ., it is easy to prove that the angle $A'VB'$ is supplementary to the diedral B-VC-D, $B'VC'$ *to* C-VD-E, and so on, around the polygon. This construction includes triangles, but picks out the polar triangle opposite to the one we have been used to consider.

The order of naming vertices as given in the preceding paragraph seems whimsical: it serves, however, to indicate the correspondence of sides and angles in the usual way when the

polygon degenerates to a triangle. A much more satisfactory method would be to name one of the polygons by its sides, thus: $abcde$ . . . and its polar polygon by its vertices $A'B'C'D'E'$ . . ., with the understanding that the radius VA' is perpendicular to the plane of the arc a at the point V.

For example, the triangle abc has for its polar triangle $A'B'C'$, such that $VA' \perp a$, $VB' \perp b$, $VC' \perp c$. Since $VA' \perp a$, and $VB' \perp b$, the plane $VA'B'$ is perpendicular to the edge of the diedral aVB; consequently the intersection of the arcs a and b (the point ab) is the pole of the arc $A'B'$, and the triangle abc is the polar triangle of $A'B'C'$.

These proofs of the usual theorems about polar triangles are, at the worst, not more difficult than the usual proofs; and they can readily be extended to polygons, if, indeed, it is not quite as convenient to give proofs about polygons in general at the start.

Spherical Excess of a Polygon

Let $ABCDE$ . . . . be any convex spherical polygon, and $a'b'c'd'e'$ . . . . its polar polygon. The planes of the sides of each of these polygons form a polyedral angle with vertex at V, the center of the sphere. What can we discover about the sum of the angles $A\ B\ C\ D\ E$ . . . ?

We may be able to use the fact, proved in the next paragraph, that $A = \pi - a'$, and the further fact that a' is the measure of one of the face angles of the polyedral angle belonging to $a'b'c'd'e'$ . . . .

Draw a plane to cut all the lateral edges of the polyedral angle of $a'b'c'd'e'$ . . ., and thereby form a pyramid. In the face of this pyramid that contains arc a', not only is it true that the angle at $V = a'$, but also that the base angles of the triangle are together equal to A; that is $A = \pi - a'$. Then, if we represent $\sum$ the sum $A + B + C + D + E +$ . . . and by p' the sum $a' + b' + c' + d' + e' +$ . . . we have the equation (1) $\sum = n\pi - p'$.

On the other hand, looking at the base of the pyramid, any angle of that plane polygon is less than the sum of the two angles of the lateral faces that come next to it; and consequently the

sum of all the angles of the base polygon, which is $(n - 2)\pi$, will be less than Σ by some undetermined amount which we may represent by X. Then we have the equation

$$(2)\ X = \Sigma - (n - 2)\pi$$

From these two equations we find

$$(3)\ X = 2\pi - p'$$

On equation (2) we may base the definition:

The spherical excess of a convex spherical polygon is the amount by which the sum of its angles exceeds the sum of the angles of a plane polygon of the same number of sides.

And we may state equation (3) as a theorem:

The spherical excess of a convex spherical polygon is equal to the difference in radians between the perimeter of its polar polygon and the circumference of a great circle.

The Area of a Spherical Triangle

Expressing the angles of a spherical triangle ABC in radians, its area by S, and representing by T_1, T_2 and T_3, respectively, the areas of the three triangles that piece S out so as to make lunes with angles A, B, and C, we have the equations:

$$S + T_1 = \frac{A}{2\pi} 4\pi r^2 = 2\ Ar^2$$
$$S + T_2 = 2Br^2$$
$$S + T_3 = 2Cr^2$$
$$3S + T_1 + T_2 + T_3 = 2(A + B + C)r^2$$
$$2S + 2\pi r = 2(A + B + C)r^2$$
$$S = (A + B + C - \pi)r^2$$

Or, representing the spherical excess of the spherical triangle by X,

$$S = Xr^2$$

The Area of a Spherical Polygon

In the convex spherical polygon $ABCDE$ . . . . draw all possible diagonals through the vertex A. This will divide the polygon into triangles having a common vertex A. Every side of the polygon except the two that pass through A, will identify a triangle by serving as its base; there will conseqently be n-2 triangles. Now if we use the same suffix for all the angles of one

triangle, we can get along with one letter for each angle; and we can write for the areas of the successive triangles—

$$S_1 = (A_1 + B_1 + C_1 - \pi)r^2 = X_1r^2$$
$$S_2 = (A_2 + C_2 + D_2 - \pi)r^2 = X_2r^2$$
$$S_3 = (A_3 + D_3 + E_3 - \pi)r^2 = X_3r^2$$

and so on; and for the area of the polygon—

$$S = S_1 + S_2 + S_3 + \ldots\ldots$$
$$S = (A + B + C_1 + C_2 + D_2 + D_3 + E_3 + E_4 + \ldots\ldots - (n - 2)\pi)r^2$$
$$S = A + B + C + D + E + \ldots - (n - 2)\pi)r^2$$
$$S = Xr^2$$

Hence the theorem:

The area of any spherical polygon is equal to its spherical excess, in radians, multiplied by the square of the radius.

The Area of a Small Circle on a Sphere

We may consider the area enclosed by a small circle on a sphere as the limit of the area of an equilateral spherical polygon of n sides, inscribed in that circle, as n is indefinitely increased. Every one of those equilateral spherical polygons will have a polar polygon, which will be inscribed in a small circle described from the nearer pole of the given circle with a polar distance $\theta + \frac{\pi}{2}$, where θ is the polar distance of the given circle. This we may call the polar circle of the given circle; and each element in the central cone determined by either circle will be perpendicular to the corresponding element in the other cone and in the same axial plane with it.

The circumference of the given circle will be $2\pi r \sin\theta$ and of the polar circle $2\pi r \cos\theta$.

For any one of the equilateral polygons inscribed in the given circle we have $S = (2\pi - p')\ r^2$ where p' is the perimeter of the polar polygon, expressed in radians.

The limit of p', as n increases, is

$$\frac{2\pi r \cos\theta}{r} = 2\pi\cos\theta$$

The area of the circle, which we may represent by Z is the limit of $(2\pi - p^1)r^2$, that is:

$$Z = 2\pi r^2 - 2\pi r^2 \cos\theta$$

$$= 2\pi\ (1 - \cos\theta)\ r^2 = 4\pi(\sin^2 \frac{\theta}{2})\ r^2$$

We have, however, as a formula for Z, independently arrived at, $Z = 2\pi rh$, where h is the altitude of the cone. It is therefore reassuring to notice that

$$h = r(1 - \cos\theta) = 2r\sin^2 \frac{\theta}{2}$$

DISCUSSION

Mathematical Clubs in the High School. In most of our teachers' meetings, and in our pedagogical magazines, we are putting forth much energy to help the average pupil to a more effective and less difficult method of mastering the requirements of mathematics. We are placing much stress on the *normal;* we are trying by special schools or groupings to help the slightly subnormal towards the "passing grade." Since we are teaching in public high schools which are supported by common taxes and since we are requiring practically all students to take algebra and geometry to secure a diploma, it is clearly our obligation to adjust our curriculum to the student of average intelligence and see that his progress through the school be unhampered by requirements which he cannot meet.

But what provision are we making for those students exceptionally gifted or particularly interested in mathematics, those who perhaps, as scientists and engineers will solve the great industrial problems of the future? Our class work, limited both by time and by the mediocre mentality of each group as a whole, affords no opportunity for the unusual pupil. To meet this need a mathematical society, named the Euclidean Club, is conducted in Scott High School. Only sophomore boys, whose grade in mathematics is A or B, and juniors and seniors who show sufficient interest to profit by membership in this organization, are admitted. A typical program consists of five numbers:

I. Theoretical Subject, as Fourth Dimension, Trisection of an Angle, Non-Euclidean Geometry, Einstein's Theory.

II. Biography of a Mathematician or Scientist: Euclid, Pythagoras, Archimedes.

III. Practical Application of Mathematics. Use of Geometry in the Construction of an Engine; Trigonometry in the Use of Projectiles.

IV. Optional subject, in the nature of a diversion, as Wall Street and the Stock Exchange; Foreign Coins.

V. Scientific or Engineering Subject: American Dyes, Construction of a Submarine, the Panama Canal.

These programs are given entirely by the members themselves. They can secure their material from books in the Scott High School Library, from magazines and from suggestions from the faculty advisor.

Occasionally an evening is given to an outside speaker, a man who has graduated from an engineering school and is engaged in a profession using mathematics. Thus, an alumnus from Massachusetts Institute of Technology spoke on the entrance requirements of his Alma Mater, courses of study, student life and also on Mechanical Engineering as a Profession. A graduate of Case School of Applied Science spoke similarly of his school and on mining engineering.

By this method the boys become familiar with the various branches of applied mathematics, with the varieties of engineering professions, their advantages and disadvantages, and with the nature of schools preparing for that work.

Through this club the boys are able to analyze themselves and to determine, in some measure, their interest in higher mathematics. The benefits of the Euclidean Club are two-fold: Some find their limitations through this introduction into higher mathematics and are thus spared a great folly. Many a bright boy who has been bored by the slow recitation has been interested by the club and has been encouraged to continue his education.

Sophia Refior

Scott High School, Toledo Ohio

RECENT ARTICLES OF INTEREST TO MATHEMATICS TEACHERS

Education for the Life of To-day: G. L. Cave, of the School Board of Gorham, N. H., School and Society, Vol. XVI., No. 402, p. 281. This gives the views of a layman on present day tendencies and is well worth reading.

Aims in American Education: Honorable Charles Evans Hughes, Secretary of State, Washington, D. C. The Journal of the National Education Association, Vol. XI., No. 7, p. 257. Mr. Hughes calls attention to the fact that the will of the people is the ultimate determining factor in education. Vocational education is not likely to suffer, but education should be broader than this. The foundation of true education should be laid in

a few studies of the highest value in self discipline. He says, " I am one of those who believe in the classical and mathematical training, and I do not think we have found any satisfactory substitute for it. . . . The important point is the insistence of concentration and thoroughness." This article should be read by every teacher.

The Uses of Algebra in Study and Reading: Edward L. Thorndike and Ella Woodyard, of Teachers' College, Columbia University, New York, N. Y. School Science and Mathematics, Vol. XXII., Nos. 5 and 6.

The Present position of the Island Universe Theory of the Spiral Nebulae: Dean B. McLaughlin. Popular Astronomy, Vol. XXX, Nos. 5 and 6. This article shows the amazing rate at which our conception of the vast sweep of the universe has grown in recent months. ALFRED DAVIS

From School and Society for August 5, 1922: Chicago's 10,000 public school teachers and principals are to receive salary increases aggregating $4,250,000 annually, effective September 1. Under the new scale the minimum annual pay of elementary school teachers, of whom there are 8,000, will be increased from $1,200 to $1,500 and the maximum from $2,000 to $2,500; the minimum of high school teachers, of whom there are 1,600 in number, from $1,600 to $2,000 and the maximum from $3,400 to $3,800; the minimum of the 268 elementary school principals from $2,500 to $3,000 and the maximum from $4,200 to $4,800, and the minimum of 23 high school principals from $3,700 to $4,300 and the maximum from $5,100 to $5,700. (ALFRED DAVIS)

Teaching Percentage with the Ruler. Motivation seems to be the most used of any pedagogical term. All teachers who class themselves as professionals *motivate* their class work. Motivation in the teaching of percentage seems to be rather difficult when we look at some of the problems in our arithmetics. For instance, here is a problem from a well known arithmetic: The number of youths of school age in a certain city is 16,767 which is 24½% of the whole number of inhabitants. What is the population of the city? It is enough to scare the pupil. To a little sixth grade pupil, the subject is unadapted. The

difficulty of apprehending such a large quantity, let alone the absurd rate of $24\frac{1}{2}\%$, is easily seen.

We must first have the subject matter of our problem close at hand and the per cents should be quantities that are easily reducible to fractions. We are after the method rather than the skill in handling large and complex quantities. The spirit for doing is killed in making the operations difficult. At the same time the sense of values cannot be readily grasped by the pupils.

Arithmetic should be made as concrete as possible. The problems of life are concrete enough. We cannot bring many real objects into the class-room. We cannot bring in bushels of grain, barrels of apples or cords of wood but we can have a ruler, a real live ruler and some yardsticks. The children can see and sense the distance of an inch, nine inches or twenty centimeters. Here is something real within the reach of every teacher. It is much more interesting to have the concrete thing than to talk about it without its being present.

In teaching percentage the writer has found no better means of motivation and of making it a live issue than to use the ruler. In the first place per cent is only another means of expressing fractional parts. It seems to be more dignified to say ten per cent than to say one-tenth. To the pupil it is only acquiring a new language and an understanding of that language in terms already learned.

They have the ruler. It is easy to see that two inches is one-fourth of eight inches, that six inches is three-fourths of eight inches, etc. Put you finger on eight inches. What part of eight inches is one inch? Two inches? Three inches? Four inches? Six inches? Seven inches? Slide the other hand along the ruler as you go.

Put your finger on twelve inches. What part of twelve inches is one inch? Two inches Three inches? etc. Keep at this drill. It never grows monotonous for the children. They like it and call for more.

Perhaps some will not respond to the question: What part of twenty-four is eighteen? Have the pupils do this. Put your finger on six inches. What part is that of twenty-four? How

many six-inch spaces are there up to eighteen inches? Answer three. Then three-fourths will come. Continue drilling by skipping about with different units as a base.

The second thing to do, if the aliquot parts have been learned by the children is to have them translate one-fourth into hundredths and then to per cent, recalling that per cent means by the hundredths. Get these translations of fractions memorized, especially the common ones which we use most.

The third thing. Using the fractional part as a bond, go over the same questions, starting thus:— Now, instead of saying what part of eight inches is one inch, we can just as well say what per cent of eight inches is one inch. Two inches? Three inches? Etc. Think it thus; one is one-eighth of eight inches; one-eighth is $12\frac{1}{2}\%$. What per cent of five inches is one inch? Two inches? Three inches? Four inches? In using the length twelve inches or any length where the parts do not result in the usual aliquot parts, write on the board the numbers you wish to use. As for twelve use, one inch, two inches, three inches, four inches, six inches, eight inches, nine inches, ten inches. It is unnecessary to teach $41\frac{2}{3}\%$, $58\frac{1}{3}\%$ and $91\frac{2}{3}\%$ in the sixth grade.

Always have each pupil follow the one reciting with their rulers. They should slide their fingers along the ruler keeping one hand at the length being used.

To teach the second type of percentage problems is no more difficult. What is one-twelfth of twelve? One-eighth of twelve? One-sixth of twelve? One-fourth of twelve? One-third of twelve? One-half of twelve? Two-thirds of twelve? Three-fourths of twelve? What is $8\frac{1}{3}\%$ of twelve? $16\frac{2}{3}\%$ of twelve? Etc. Have the pupils find the distances on their rulers.

The last type of percentage problems—ratio, and percentage given to find base, is hard to get over in the ordinary way but with the aid of the ruler it is very easy. If two inches is one-fourth of the distance, what is the whole distance? Etc. Two inches is 25% of what distance? Three inches is 25% of what distance? Etc. Starting with one inch, using 25% go the whole length of the ruler or yardstick. Use at first all the per cents whose fractional equivalents have one for their numerator, as

$8\frac{1}{3}$%, $12\frac{1}{2}$%, $16\frac{2}{3}$%. Drill these thoroughly before going on.

Start with $66\frac{2}{3}$% or 75% next. $66\frac{2}{3}$% is two-thirds. Four is $66\frac{2}{3}$% of what distance? Four is two-thirds of the distance. One-third of the distance must be two inches. The whole distance must be six.

The children should at all times indicate with their fingers the distances. This is very important. By doing this the children group and fix in their minds the operations. The contacts are three-fold,—intellectual, visual and motor. Movement aids in interest. When the pupils have mastered the language and the operations in all types of problems in percentage, one may give problems whose answers are mixed numbers.

NATHAN R. HOWELL,
East Moriches, N. Y.

NEWS AND NOTES

At three of the meetings of our mathematics club this summer we discussed matters relating to the report of the National Committee. First, Professor George E. Myers of the University of Chicago School of Education gave a general outline of the work of the committee, and this was supplemented in the discussion by Professor H. E. Slaught. At the second meeting we had Professor A. R. Crathorne of the University of Illinois, who is a member of the committee and who spoke with authority in reference to the purposes and ideals of the work of the committee, also giving considerable details. At the third meeting we had Dr. Eula Weeks of the Cleveland High School, St. Louis, who is also a member of the National Committee, and who spoke with reference to the committee's report on college entrance requirements. Miss Weeks was a member of the special commission appointed by the College Entrance Examination Board to report to them on the college entrance examination requirements; hence she was in position to speak definitely concerning the interrelations of these two committees. We had a large attendance at all of these meetings, approximately one hundred summer students, most of whom are teachers and many of whom doubtless are already members of the Council. H. E. SLAUGHT

The program of the Mathematics Section of the Central Association of Science and Mathematics Teachers to be given in Chicago, December 1 and 2 consists of: *Organization of Secondary Mathematics,* Professor Walter W. Hart, School of Education, University of Wisconsin, Madison, Wis.; *Consistency in Grading Mathematics Papers,* Professor E. J. Moulton, Northwestern University, Evanston, Ill.; *The Slide Rule,* W. W. Gorsline, Crane Junior College, Chicago, Ill.; *Preparation of Teachers of Mathematics for Junior High Schools,* Professor J. R. Overman, State Normal College, Bowling Green, Ohio; *Inspection of Some Old Mathematical Manuscripts from Armour Institute and other Sources,* M. J. Newell, Township High School, Evanston, Illinois.

W. G. Gingery, of Shortridge High School, Indianapolis, is chairman.

The program of the Mathematics Section of the Illinois High School Conference, held at Urbana, November 24, 1922, consisted of:

1. *Unified Mathematics,* E. R. Breslich.
2. *Socializing the Mathematics,* Everett W. Owen.
3. *The Use of the Slide Rule in Secondary Schools,* W. W. Gorsline.
4. *Teaching Secondary Mathematics,* Miss Edith Atkins.

Professor W. T. Felts, of Carbondale is chairman of the section.

The seventeenth regular meeting of the Association of Mathematics Teachers of New Jersey was held at Rutgers College, New Brunswick, N. J., on October 28th, in conjunction with the annual New Jersey State High School Conference. Seventy members were in attendance. The program was as follows:

Some Aims in the Teaching of Mathematics, Mr. R. H. Rivenburg, Peddie Institute.

The Cyclic Quadrilateral, Professor Richard Morris, Rutgers College.

When is an Examination Not an Examination?, Miss Josephine Emerson, Kent Place School, Summit.

Note on Simultaneous Quadratic Equations, Mr. Harrison E. Webb, Central High School, Newark.

A committee on examinations and standard tests in Mathematics, under the chairmanship of Howard F. Hart, of Montclair High School, was appointed by the president. This committee will consider not only the various forms of proposed standard tests, but also the established types of examination, such as the New York State Regents' Examination and those of the College Entrance Examination Board. The personnel of the committee will be announced later. The Officers of the Society for the present year are:

President, P. W. Averill, Battin High School, Elizabeth.

Vice-President, H. F. Hart, Montclair High School.

Secretary-Treasurer, Andrew S. Hegeman, Central High School, Newark.

Council Members, Professor Charles O. Gunther, Professor Richard Morris, Dean Henry B. Fine, Dr. Fletcher Durell, Miss Josephine Emerson, Professor C. R. MacInnes.

THE RESEARCH DEPARTMENT

For this month the Department proposes: *The Problem of Differentiated Curricula, or How to Meet the Needs of Pupils Who Have Been Classified on the Basis of General Intelligence.*

Many schools are now classifying their children on the basis of general intelligence, presumably to get children in groups that will be able to do more nearly the same type or equal amounts of work. For this purpose, a combination of the following are used: (1) general intelligence tests, (2) marks received at the close of the preceding term, (3) the judgment of teachers, (4) the scores on standard achievement tests, and (5) the scores on informal tests in arithmetic, algebra and geometry made up by the teacher. Whatever the method of classifying may be, we may assume that the groups obtained in schools of considerable size differ markedly from each other in abilities and needs.

Our interest as teachers of mathematics is what happens to these children as far as concerns mathematics after they have been classified. Are the different groups being taught different textbooks? Are higher standards of attainment expected of the higher groups? Does the class with the highest ability study Algebra? Do the lowest groups emphasize commercial and industrial phases of mathematics? Does an introductory course in mathematics meet the needs of all groups? Do all study the same materials but move at different rates? These are some of the subsidiary questions of the general problem which teachers of mathematics must face in increasing numbers.

It would seem that the readers of this journal could render a valuable service to each other by mobilizing their experiences. With this in mind, the following data are requested:

(1) A statement telling how children are classified in your school. Include here the length of time that the present policy has been in effect.

(2) A description of the courses for the different groups. When possible, mention texts and pages covered in a year by each group.

(3) A report of differences in teaching methods and standards required in the different classes.

It may have been noted in the news section of this number that the Chicago Mathematics Club has chosen this problem for one of its meetings. This suggests that the mathematics clubs of other cities may also want to assemble materials and give the readers of the Mathematics Teacher the evidence considered and the conclusions reached. RALEIGH SCHORLING

NEW BOOKS

Plane Geometry. By C. Addison Willis. P. Blakiston's Son and Company, Philadelphia. Pp. 301.

General Mathematics, Book II, By William David Reeve. Ginn and Company, Pp. 446.

Junior High School Mathematics, Third Book. By E. H. Taylor and Fiske Allen. Henry Holt and Company, New York. Pp. 155.

Plane Trigonometry. By Professor Leonard Eugene Dickson. Benj. H. Sanborn and Company, New York. Pp. 176+35.

Elementary Calculus. By Frederick S. Woods and Frederick H. Bailey. Ginn and Company, New York and Chicago, Pp. 318.

Junior High School Mathematics, Book Two. By Walter W. Hart. D. C. Heath and Company. Pp. 256.